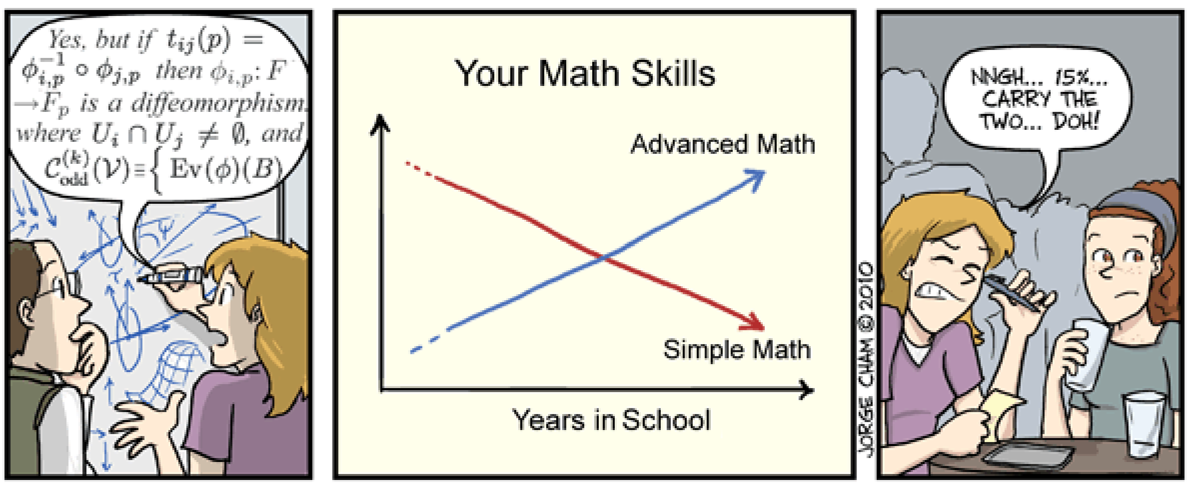
MANAGERIAL ECONOMICS

ECNS 309 COURSE NOTES



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Any errors in this document are the responsibility of the author. Corrections and comments regarding any material in this text are welcomed and appreciated. The material in this document is intended to be supplementary (not as a substitute) to attending lectures regularly.

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Chapter 1: Introduction

Economics is the study of individuals' choices, decisions, and behaviors. One of the primary values that economists can contribute is by applying mathematical and statistical tools to measuring these choices, decisions, and behaviors and interpreting those measures within the context of comprehensible economic theory. Managerial economics is one of the first courses in which this intersection between analytical tools and economic theory begins to become apparent, and you are tasked with developing these skills. Ultimately, successful management of any operation—regardless of size—depends on efficiently making effective decisions about the operation and understanding how your operation and your customers will respond to these decisions.

The diagram in Figure 1.1 describes the role of managerial economists within an operation. The figure helps demonstrate the types of questions managerial economists are asked, the set of tools available to managerial economists for answering those questions, and the result of that analysis.

Making optimal decisions when an operation faces numerous constraints is one of the primary goals for a managerial economist. This often involves evaluating numerous trade-offs, costs, and benefits in order to reach a balance among the numerous aspects that a manager must consider. Figure 1.2 provides a visual example of the aspects that a firm must consider and the balance that the firm must seek to achieve. The figure shows that for some goals, a firm may be below an optimal level of achieving those goals, while for others, the firm may be above the optimal level. As managers, your objective is to allocate (or reallocate) resources in order to be as close as possible to the optimal level for all of an operation's established goals.

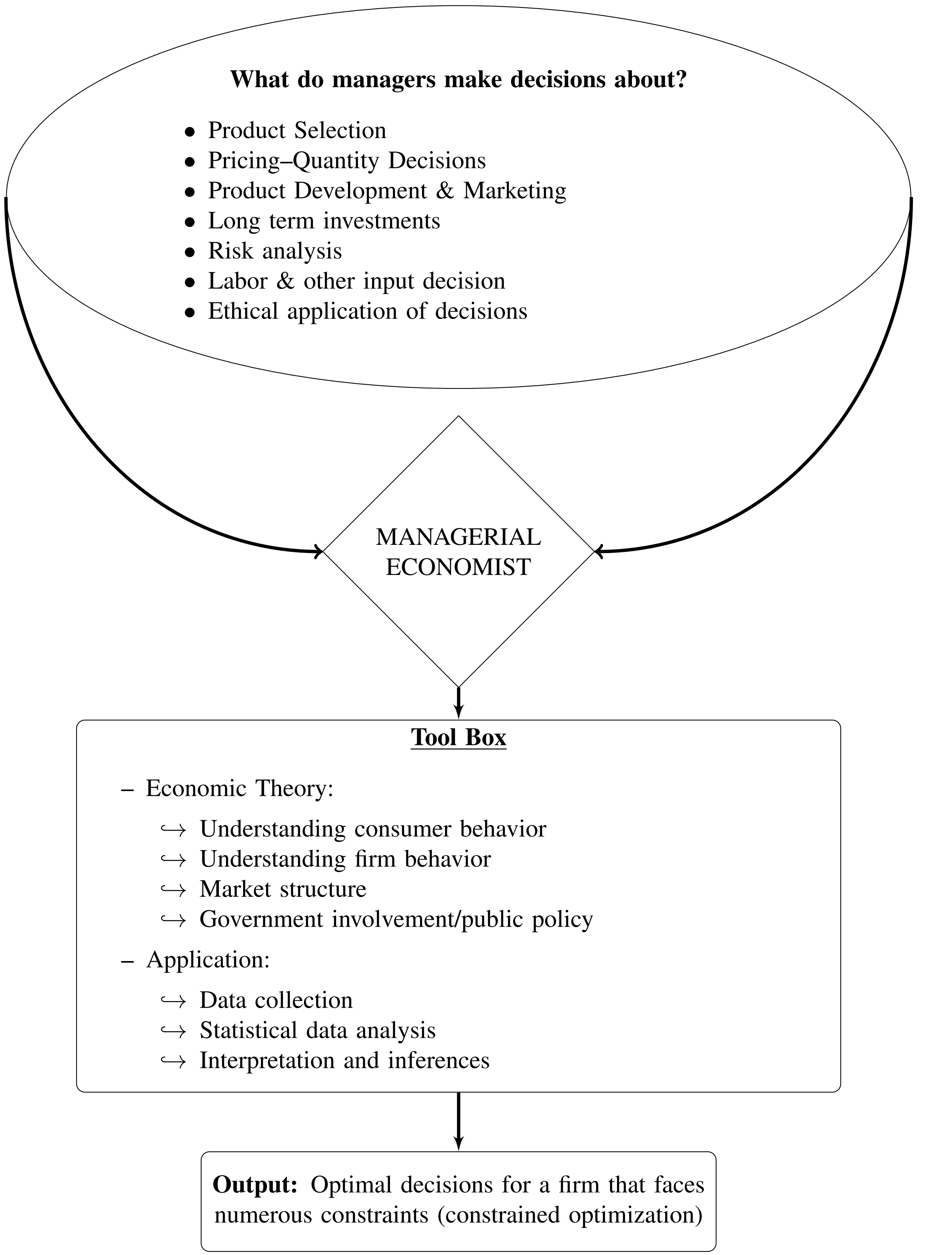


Figure 1.1: The Role of a Managerial Economist in an Operation

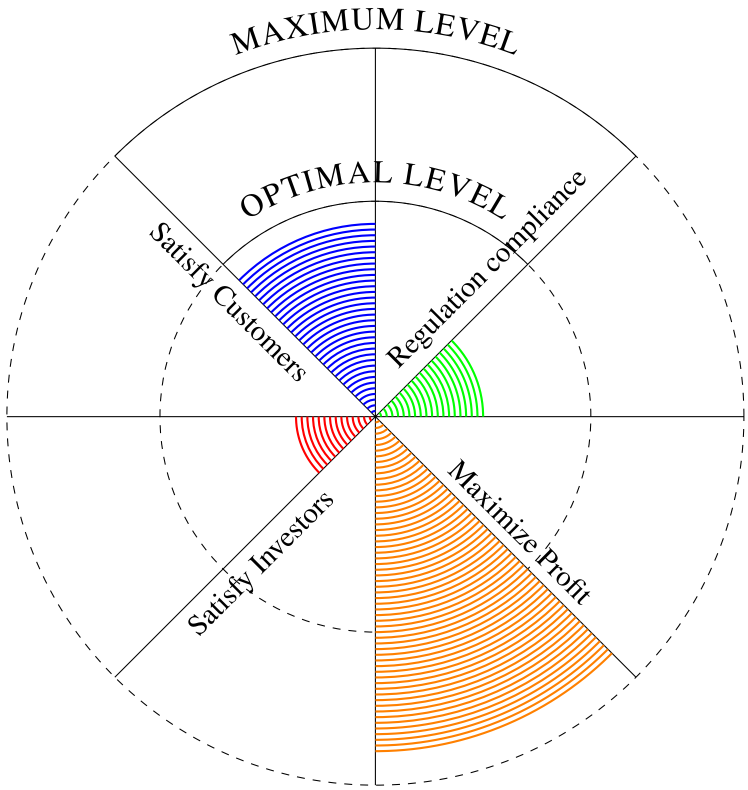


Figure 1.2: Visualizing a Firm's Goals and Optimal Attainment Levels for those Goals

The evaluation is the critical skill that managerial economists must have to ensure that their operation is successful in the presence of numerous obstacles. That is, they must both see the "big picture" and have the skills to measure deficiencies and effectively and efficiently adjust operations to overcome those weaknesses.

How does this course fit in?

During the semester, we will focus on techniques to understand potentially sub-optimal conditions face by an operation, learn the tools to measure factors that may contribute to those conditions, and apply those tools to make decisions for overcoming weaknesses. The course structure is divided into four broad categories.

1. **Theoretical analysis of consumers**
   - Tools: economic theory and calculus.
   - Application:
     - Demand-side analysis—knowing your market and knowing your consumer.
     - Marginal analysis—evaluating the results of small changes to the operation.
2. **Empirical analysis**
   - Tools: basic statistics and regression analysis.
   - Application: using real-world data to estimate demand and other important relationships and make forecasts.
3. **Theoretical analysis of firms**
   - Tools: economic theory and calculus.
   - Application: analyzing production decisions under constraints.
4. **Risk and uncertainty**
   - Tools: economic theory and statistics.
   - Application:
     - Evaluate decisions when outcomes are uncertain.
     - Develop approaches for evaluating trade-offs of uncertain outcomes.
     - Assess risk, factors contributing to the risk, and methods for adjustments.

Chapter 2: Math Review

While not a course in mathematics, managerial economics requires students to be comfortable and proficient in algebra and calculus. A review of the most relevant topics is as follows.

2.1 Exponents

Exponents are a method for representing multiple instances of multiplication. For example,

$$x \cdot x \cdot x = x^3$$

In general, when x is multiplied n times by each other, we denote that as x^n. We denote x to be the *base* and n as the *exponent*.

The same principle can be applied to more complex repeated values. For example,

$$(1+z) \cdot (1+z) \cdot (1+z) = (1+z)^3$$

If we let $(1+z) = x$, then we can directly apply the more basic example above to obtain the shortened notation $(1+z)^3$.

2.1.1 Negative exponents

For any negative exponent, we can re-write the value as the inverse of the base and the negative value of the exponent. That is,

$$x^{-n} = \frac{1}{x^n}$$

Similarly,

$$x^n = \frac{1}{x^{-n}}$$

2.1.2 Operations with exponents

First, consider multiplying values that have the same base but different exponents. In this case, we add (or subtract) the exponents and maintain the same base.

$$
\begin{aligned}
x^n x^m &= x^{n+m} \\
x^n x^{-m} &= x^{n-m} \\
\frac{x^n}{x^m} &= x^n x^{-m} = x^{n-m} \\
\frac{1}{x^n x^m} &= x^{-(n+m)}
\end{aligned}
$$

Second, consider multiplying factors with a different base but a common exponent. In this case, we can combine and multiply the exponents.

$$x^n z^n = (xz)^n$$

Third, when exponents are raised to a power, they can be multiplied together.

$$
\begin{aligned}
(x^n)^m &= x^{nm} \\
(x^n)^{\frac{1}{m}} &= x^{n/m}
\end{aligned}
$$

2.1.3 Exponent roots

Exponent roots are typically represented by exponent fractions.

$$
\begin{aligned}
\sqrt{x} &= x^{1/2} \\
\sqrt[n]{x} &= x^{1/n} \\
\sqrt[n]{x^m} &= (x^m)^{\frac{1}{n}} = x^{m/n} \\
\sqrt[n]{\frac{1}{x}} &= \frac{1}{x^{1/n}} = x^{-1/n} \\
\sqrt[n]{\frac{z^m}{x^v}} &= z^{m/n} x^{-v/n}
\end{aligned}
$$

2.1.4 Limits

Often, we will be interested in evaluating functions as we increase or decrease the value of an exponent to some large, undefined magnitude. For example, if x^n, then the following holds.

If $x = 0$ and $n \neq 0$,	then $x^n = 0$
If $x \neq 0$ and $n = 0$	then $x^n = 1$
If $x < 0$ and $-1 < n < 1$	then $x^n =$ undefined
If $0 < x < 1$ and $n \to \infty$	then $x^n = 0$
If $x > 1$ (or $x < -1$) and $n \to \infty$	then $x^n \to \infty (x^n \to -\infty)$
If $0 < x < 1$ and $n \to \infty$	then $x^{-n} \to \infty$
If $x > 1$ (or $x < -1$) and $n \to \infty$	then $x^{-n} = 0$

2.2 Logarithms

Logarithms are the complement to exponents. We will often use logarithms to simplify (i.e., linearize) complex exponential functions.

Let $y = x^n$. Then,

$$n = \log_x y$$

where x is the base. Assume that if $n = \log y$, then we assume a log base 10 (i.e., $x = 10$). If $n = \ln y$, then we assume a log base e (i.e., a natural log). We will most frequently use the natural log notation.

2.2.1 Logarithm properties

The following properties are useful to know and commit to memory.

$$
\begin{aligned}
\ln(mn) &= \ln m + \ln n \\
\ln(m/n) &= \ln m - \ln n \\
\ln(m^n) &= n \cdot \ln m \\
\ln(m+n) &\neq \ln m + \ln n \\
\ln 1 &= 0 \\
\ln n, \text{ where } n \leq 0 &= \text{undefined} \\
\ln n, \text{ where } 0 < n < 1 &< 0 \\
\ln n, \text{ where } n > 1 &> 0
\end{aligned}
$$

2.3 Miscellaneous

Along with understanding exponential and logarithmic notation, there are a number of other formulas and notations that will be used frequently throughout the course.

2.3.1 Quadratic formula

The classic formula is useful to determining roots. That is, in the equation $ax^2 + bx + c = 0$, solving for the values of x at which the equality will hold.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2.3.2 Factorial

The general factorial notation is, $n! = 1 \cdot 2 \cdot 3 \cdots n$. For example,

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

2.3.3 Summation notation

Summation notation is one of the most useful tools for increasing efficiency in denoting long, repeated series of operations. The general summation notation is,

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_n$$

For example,

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

Any scalars inside of a summation can be brought out of the summation. For example, suppose c is a scalar. Then,

$$\begin{aligned} \textstyle\sum_{i=1}^{n} cx_i &= cx_1 + cx_2 + \ldots + cx_n \\ &= c(x_1 + x_2 + x_3 + \ldots + x_n) \\ &= c\textstyle\sum_{i=1}^{n} x_i \end{aligned}$$

That is, the scalar c does not depend on i and can, therefore, be taken outside of the summation. Conversely, x does depend on i (as indicated by the subscript) and must remain inside of the notation.

Derivatives

Definition and Notation

If $y = f(x)$ then the derivative is defined to be $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

If $y = f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = Df(x)$$

If $y = f(x)$ all of the following are equivalent notations for derivative evaluated at $x = a$.

$$f'(a) = y'\big|_{x=a} = \frac{df}{dx}\bigg|_{x=a} = \frac{dy}{dx}\bigg|_{x=a} = Df(a)$$

Interpretation of the Derivative

If $y = f(x)$ then,

1. $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x-a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$.
3. If $f(x)$ is the position of an object at time x then $f'(a)$ is the velocity of the object at $x = a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(cf)' = cf'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(fg)' = f'g + fg'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c) = 0$
6. $\frac{d}{dx}(x^n) = nx^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

   This is the **Chain Rule**

Common Derivatives

$\frac{d}{dx}(x) = 1$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(a^x) = a^x \ln(a)$

$\frac{d}{dx}(\mathbf{e}^x) = \mathbf{e}^x$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x},\ x > 0$

$\frac{d}{dx}(\ln|x|) = \frac{1}{x},\ x \neq 0$

$\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln a},\ x > 0$

Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}\left(\left[f(x)\right]^n\right) = n\left[f(x)\right]^{n-1} f'(x)$
2. $\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\mathbf{e}^{f(x)}$
3. $\frac{d}{dx}\left(\ln\left[f(x)\right]\right) = \frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}\left(\sin\left[f(x)\right]\right) = f'(x)\cos\left[f(x)\right]$
5. $\frac{d}{dx}\left(\cos\left[f(x)\right]\right) = -f'(x)\sin\left[f(x)\right]$
6. $\frac{d}{dx}\left(\tan\left[f(x)\right]\right) = f'(x)\sec^2\left[f(x)\right]$
7. $\frac{d}{dx}\left(\sec\left[f(x)\right]\right) = f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\frac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right) = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Higher Order Derivatives

The Second Derivative is denoted as $f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2}$ and is defined as $f''(x) = (f'(x))'$, *i.e.* the derivative of the first derivative, $f'(x)$.

The n^th^ Derivative is denoted as $f^{(n)}(x) = \frac{d^n f}{dx^n}$ and is defined as $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$, *i.e.* the derivative of the (n-1)^st^ derivative, $f^{(n-1)}(x)$.

Implicit Differentiation

Find y' if $\mathbf{e}^{2x-9y} + x^3y^2 = \sin(y) + 11x$. Remember $y = y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). After differentiating solve for y'.

$$\mathbf{e}^{2x-9y}(2-9y') + 3x^2y^2 + 2x^3yy' = \cos(y)y' + 11$$
$$2\mathbf{e}^{2x-9y} - 9y'\mathbf{e}^{2x-9y} + 3x^2y^2 + 2x^3yy' = \cos(y)y' + 11 \quad \Rightarrow \quad y' = \frac{11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2}{2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)}$$
$$\left(2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)\right)y' = 11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2$$

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x = c$ is a critical point of $f(x)$ provided either **1.** $f'(c) = 0$ or **2.** $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x) < 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x = c$ is a inflection point of $f(x)$ if the concavity changes at $x = c$.

2.5 A Review of Derivatives and Applications

The application of derivatives is one of the fundamental aspects of economics. Economists evaluate marginal changes in behaviors and outcomes, and derivatives are the mathematical tools for making these evaluations. For example, economists are able to assess trade-offs by understanding the marginal costs and marginal benefits of action. Firms can determine the return to an investment by deriving the marginal product of an input. In general, derivatives help us answer the question: If some behavior or variable x changes by some small amount, how much of a change will occur in an outcome y?

Consider the function $y = f(x)$. The general form of the derivative is $\frac{dy}{dx} = f'(x)$. The interpretation is the change in the value of the variable y (change in "rise"), given that there is a change in the value of the variable x (change in "run"). The intuition behind interpreting the ratio $\frac{dy}{dx}$ is that we want to measure the response in our variable of interest y to "a little bit of change" in the variable x.

Example 2.5.1 Intuition underlying derivatives

Consider the function $y = x^2$ and find $\frac{dy}{dx}|_{x=3}$. That is, find $\frac{dy}{dx}$ at $x = 3$.

x	y	dx (from $x = 3$)	dy (from $y = 9$)	$\frac{dy}{dx}$
3	9	—	—	—
4	16	+1	+7	7/1 = 7
3.10000	9.61000	+0.10000	+0.61000	0.61000/0.10000 = 6.100
3.01000	9.06010	+0.01000	+0.06010	0.06010/0.01000 = 6.010
3.00100	9.00601	+0.00100	+0.00601	0.00601/0.00100 = 6.001

As $dx \to 0$ (gets smaller), $\frac{dy}{dx}$ converges to 6. If we were to use calculus, $\frac{dy}{dx} = 2x$. Therefore, $\frac{dy}{dx}|_{x=3} = 6$. That is, for a very small change in x (i.e., dx is very close to 0), $\frac{dy}{dx}|_{x=3} = 6$.

It is important to note that because dy and dx are assumed to be very small changes, then $(dy)^2 \approx (dx)^2 \approx (dy \cdot dx) \approx 0$. That is, any product of the terms, will be even smaller, and we will assume that the value of that product is essentially zero.

Example 2.5.2 Intuition underlying derivatives, another perspective

Again consider the function $y = x^2$. Now, suppose that we add small changes to y and x and solve for $\frac{dy}{dx}$.

$$\begin{aligned} y + dy &= (x + dx)^2 \\ y + dy &= x^2 + 2x \cdot dx + (dx)^2 \\ x^2 + dy &= x^2 + 2x \cdot dx \\ dy &= 2x \cdot dx \\ \frac{dy}{dx} &= 2x \end{aligned}$$

Note that in the third step, we used the fact that $y = x^2$ and that $(dx)^2 \approx 0$. Therefore, it is apparent that the calculus solution is equivalent to the algebra solution above.

As managerial economists, you will use derivatives to determine the extreme points on various demand, supply, cost, revenue, and profit functions, among others. This will help answer questions such as:

- At what price are we maximizing revenue or profit function?
- At what amount of inputs or prices are we minimizing costs?
- At what level of inputs are we maximizing production?

Figure 2.1 provides a visual example of these types of objectives. In each figure, the goal is to find the level of x at which the value of the function is either the maximum or the minimum. As shown in the figure, this value of x is determined at the point where the line tangent to the curve has a slope of 0; that is, $\frac{dy}{dx} = 0$.

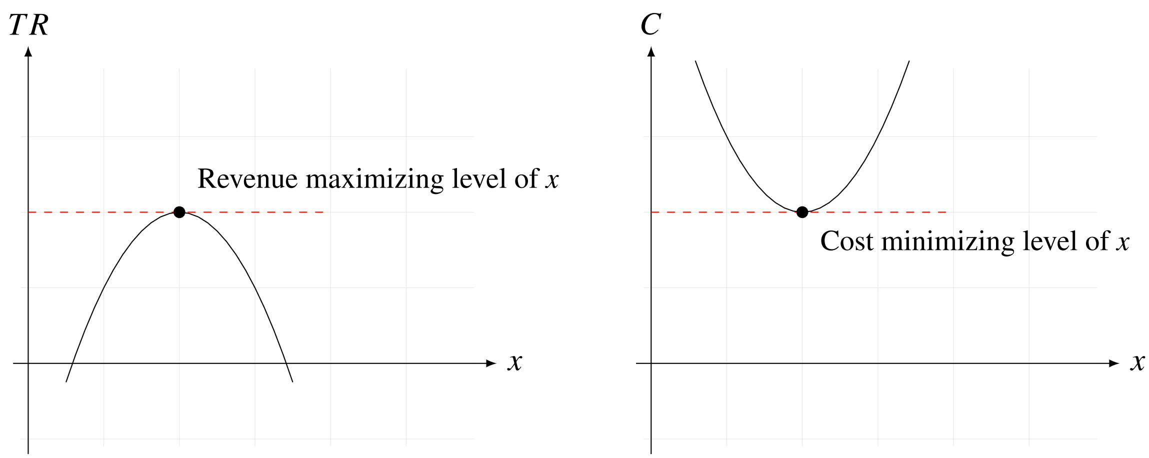


Figure 2.1: Identifying maximum and minimum points of a function

In general, consider some function $y = f(x)$. To determine the value of x that maximizes or minimizes y:

1. Take the first-order derivative of y with respect to x, and set this equation equal to zero; that is, $\frac{dy}{dx} = 0$.
2. Solve for the maximizing/minimizing level of x. We will use the notation x_* to specify that this value is maximizing/minimizing.
3. Determine whether the function y is maximized or minimized at x_* by taking the second-order derivative, $\frac{d^2y}{dx^2}$.
   - If $\frac{d^2y}{dx^2} > 0$, then x_* minimizes the function y.
   - If $\frac{d^2y}{dx^2} < 0$, then x_* maximizes the function y.

Intuition: The second-order derivative provides insights about the direction in which you will move on the function if you were to move away from x_*. That is, if $\frac{d^2y}{dx^2} < 0$, then moving away from x_* implies that you will be moving down the function to lower values of y. If this is the case, then you know that there are no other values of x_* at which you can obtain a higher value of y. Similarly, if $\frac{d^2y}{dx^2} > 0$, then moving away from x_* implies that you will be moving up the function to higher values of y. If this is the case, then you know that there are no other values of x_* at which you can obtain a lower value of y.

Example 2.5.3 Identifying the maximum/minimum value of y

Consider the function $y = 5 + 2x - 0.5x^2$. Determine the value of x that maximizes or minimizes this function and show whether this value maximizes or minimizes y.

First, find the optimal value of x by taking the first-order conditions, setting them equal to zero, and then solving for x_*

$$\begin{aligned} \frac{dy}{dx} &= 2 - x = 0 \\ x_* &= 2 \end{aligned}$$

To determine whether x_* maximizes or minimizes y, determine the second-order conditions.

$$\frac{d^2y}{dx^2} = -1$$

Therefore, because $\frac{d^2y}{dx^2} < 0$, x_* maximizes the function y. At $x_* = 2$, $y = 7$.

Example 2.5.4 Identifying multiple maximums/minimums

Consider the function $y = 12 - x - x^2 + 0.1x^3$. Find the value of x that maximizes the function y.

$$\begin{aligned} \frac{dy}{dx} &= -1 - 2x + 0.3x^2 = 0 \\ x_* &= \frac{2 \pm \sqrt{-2^2 - 4(-1)(0.3)}}{2(0.3)} = (-0.47, 7.13) \end{aligned}$$

In this example, there are two values of x at which the slope of y is $\frac{dy}{dx} = 0$. So, where is the maximum?

$$\begin{aligned} \frac{d^2y}{dx^2} &= -2 + 0.6x \\ \frac{d^2y}{dx^2}|_{x=7.13} &= 2.28 \\ \frac{d^2y}{dx^2}|_{x=-0.47} &= -2.28 \end{aligned}$$

Because $\frac{d^2y}{dx^2}|_{x=7.13} > 0$, then $x_* = 7.13$ minimizes the function. Conversely, because $\frac{d^2y}{dx^2}|_{x=-0.47} < 0$, the maximum value of y is at $x_* = -0.47$.

End of Chapter Practice Problems

1. Consider two cargo ships, A and B, moving toward the same port city. The ships are 350 miles apart. Ship A is moving at 40 mph and ship B at 30 mph. A fish swims from one ship to another at a speed of 60 mph. The fish moves back and forth until both ships meet at the port. What is the total distance traveled by the fish?

2. Assume that you invest $200 in a mutual fund. The rate of growth of the mutual fund is 5% per year. When will your investment yield a $1,000 return? (Assume either annual or continuous compounding).

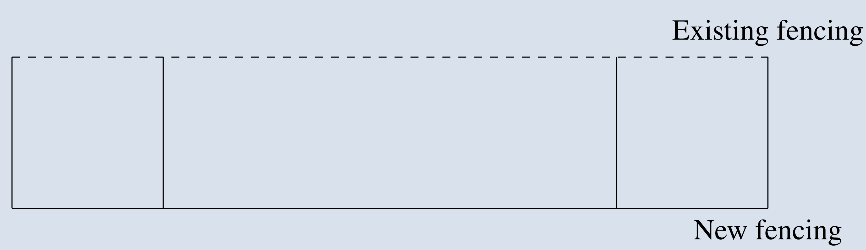
3. Suppose that you are a swim coach and have two swimmers. There are two races—sprint and long distance—for which you want to assign each of the swimmers. Swimmer A swims at a rate of $100t^{0.5}$ and swimmer B at a rate of $20t$, where t is time in seconds.

   (a) Which swimmer should be the sprinter and which should be the long distance participant?

   (b) How long (in time, t) does the race have to be to result in a tie?

   (c) If swimmers A and B are in the same race, at what time t will swimmer A be most ahead of swimmer B?

4. Suppose that you are a rancher who would like to put up a new set of pens for your animals. To minimize material costs, you decide to use a part of a fence that already exists. You have enough money to purchase an additional 300 feet of fencing and construct the fence with the following layout (dashed line is existing fencing and solid lines are new fencing):



   What is the configuration of the width and length of the new fencing that will maximize the area of the pens?

Chapter 3: Statistics Primer

Statistics are everywhere! Moreover, we use them every single day, whether you know it or not. All of our decisions are based on evaluating trade-offs, and for most of these decisions, we must make a choice prior to knowing the outcome of our the decision. As a result, our choices and behaviors are made based on an internalized assessment of the likely outcome. This outcome may not actually occur, but since we have some information about the chance that it will or will not happen, the decision is based on that information.

For example, suppose you drive to grab dinner with a friend at a local pizza place. As you near an intersection, the stoplight turns yellow. Do you try to slow down and stop before the light turns red? Do you keep going at the same speed (or accelerate) to pass the intersection before the yellow light turns red? What are the factors that you might consider when making this split-second decision?

- The likelihood that the light turns red while you are still at the intersection.
- The likelihood that there is a police officer nearby?
- The likelihood that the police officer will stop you and issue a citation for running a red light?
- The likelihood that you will get into an accident?
- The likelihood that you will be late for dinner as a result of stopping at the light?
- The likelihood that you will be late for dinner as a result of going through the light and getting stopped by a police officer?

You can certainly think of many other questions and decisions about which there is uncertain outcomes. As managers, you will be tasked to make decisions about many aspects for which outcomes are uncertain. For example, the response in quantity demanded to a change in price is not certain. It is likely that consumers will increase their purchasing behaviors in response to lower prices, but what will be the magnitude of that response?

3.1 Uncertainty and Risk Basics

Uncertainty is the idea that there is no guarantee of a particular outcome occurring. This is relevant across almost all economics and management topics because you deal with people's behaviors. However, despite not having an absolutely certain knowledge of what will happen if you, as a manager, make a decision, you are still responsible for understanding and attempting to account for the many possibilities that may occur.

Recognizing and account for uncertainty is based on your evaluation of the probability that certain outcomes will occur. For example, consider a chess or checkers match. A beginner might simply think of only his own next move. A more advanced player will think of her next move and also the possible responses that her opponent would have for that move. An expert considers patterns of his and his opponent's previous moves, the numerous potential moves that the expert can make to increase his chances of winning, the probability of the numerous responses that his opponent can make to those moves, and the possible consequences of all of those combinations.

The latter is what a good managerial economist is expected to do.

When approaching a problem that requires a consideration of uncertainty, there are three steps that you will need to take for making optimal decisions under uncertainty.

1. Evaluate the economic theory and models for describing the uncertainty and potential outcomes.
2. Use statistical methods to use historical data and information to determine the parameters for your economics models.
3. Assess your results and potentially incorporate more subjective, experience-based tweaks to improve your evaluations.

Each of the steps has advantages and disadvantages and so it is important to use the combination of all of those steps to develop an optimal strategy.

3.1.1 Expected values

The expected value will be the theoretical workhorse for assessing outcomes under uncertainty. The expected value is defined as:

$$E[V] = Prob(\text{Outcome}) \times (\text{Value of outcome})$$

That is, it is product of the probability that a particular outcome will occur and the value of that outcome. For example, if tomorrow's forecast calls for a 50% chance of rain that will result in 1 inch of precipitation, the *expected value* of tomorrow's precipitation is,

$$E[\text{Precip}] = (0.50) \times (1\text{ in}) = 0.50\text{ in}$$

Of course, if we are considering more than one outcome, we can simply use a cumulative method:

$$\begin{aligned} E[V] &= Prob(\text{Outcome 1}) \times (\text{Value of outcome 1}) + \\ &\quad Prob(\text{Outcome 2}) \times (\text{Value of outcome 2}) + \\ &\quad Prob(\text{Outcome 3}) \times (\text{Value of outcome 3}) + \\ &\quad \ldots \end{aligned}$$

Example 3.1.5 Calculating Expected Values

Suppose that you are evaluating the expected value of your income working at a food cart near a baseball stadium. Since you have to purchase your food cart permit ahead of time, you don't know what the conditions of the market will be at the time of a baseball game. For example, if the baseball team is doing well, the attendance might be much higher resulting in a higher return. Alternatively, if there is poor weather or the team is in last place, attendance might be very low and you might end up losing money if you are not able to offset the costs for your food cart permit.

Assume that the probability of a well-attended game is 30% and that, in this situation, your net return would be $100. The probability of a poorly attended game is 70% and your resulting net return would be –$20. What is your expected net return?

$$\begin{aligned} E[NR] &= Prob(\text{Good attendance}) \times (\text{NR if good attendance}) + \\ &= Prob(\text{Poor attendance}) \times (\text{NR if poor attendance}) \end{aligned}$$

$$\begin{aligned} E[NR] &= (0.30) \times (\$100) + (0.70) \times (-\$20) \\ &= \$16 \end{aligned}$$

The expected net return is a positive $16. From a managerial decision standpoint, the positive expected net return suggests that you should purchase the permit and sell food from your cart near the stadium.

Example 3.1.6 Using Expected Values to Make Investment Decisions

You are managing a firm that is deciding how to re-invest their profits. The firm is choosing among three investment strategies: (1) expand the production capacity by 50%; (2) expand the production capacity by 10%; (3) invest in a safe low volatility U.S. treasury bill (T-bill).

The re-investment will be done several months from now and you don't know what the general macroeconomy will look like. There is an 80% chance that the economy will be in a boom period and 20% that it will be in a recession. Under each macroeconomic scenario, your net returns from the three different investments are as follows:

	Boom	**Recession**
NR_1	\$50,000	–\$60,000
NR_2	\$16,000	\$11,000
NR_3	\$2,000	\$2,000

What should the optimal re-investment strategy be for this firm?

To determine the optimal strategy, it is necessary to determine the expected net returns for each strategy:

$$\begin{aligned} E[NR_1] &= Prob(\text{Boom}) \times (NR_1 \text{ if boom}) + Prob(\text{Recession}) \times (NR_1 \text{ if recession}) \\ &= (0.80) \times (\$50000) + (0.20) \times (-\$60000) \\ &= \$28000 \end{aligned}$$

$$\begin{aligned} E[NR_2] &= Prob(\text{Boom}) \times (NR_2 \text{ if boom}) + Prob(\text{Recession}) \times (NR_2 \text{ if recession}) \\ &= (0.80) \times (\$16000) + (0.20) \times (\$11000) \\ &= \$15000 \end{aligned}$$

$$\begin{aligned} E[NR_3] &= Prob(\text{Boom}) \times (NR_3 \text{ if boom}) + Prob(\text{Recession}) \times (NR_3 \text{ if recession}) \\ &= (0.80) \times (\$2000) + (0.20) \times (\$2000) \\ &= \$2000 \end{aligned}$$

There are two insights from these results:

1. The company should never invest in a T-bill. The expected return from either of the expansion strategies is higher than the expected return from the T-bill.
2. Despite the potential for a very large loss from strategy 1, the expected return of expanding by 50% is higher than doing a smaller expansion. This result is due to the fact that the likelihood of a recession is so much smaller than a boom. However, it is important to recognize that the low probability does not preclude a recession; it simply implies that the recession is less likely.

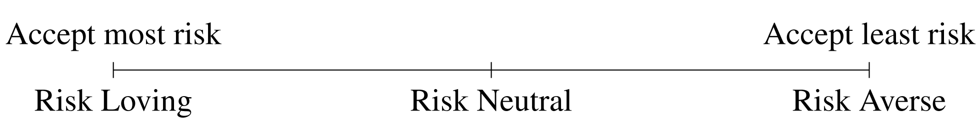
3.1.2 Risk Aversion

The most straightforward representation of the expected value, $E[V] = Prob(\text{Outcome}) \times (\text{Value of outcome})$, represents an assumption that the individual or firm for which the expected value is being calculated is *risk neutral*.

More generally, risk aversion is the concept that different individuals and firms may respond differently to uncertainty and risk. Some firms may be more risk averse and place a greater penalty for adverse outcomes that have a higher likelihood of occurring. For example, these firms would rather accept a lower net return but which has a higher likelihood of occurring, than take a chance that could result in a higher net return but could also result in a lower (or negative) net return.

Other individuals and firms may be less risk averse or even more risk loving. This group is willing to take on more uncertainty as long as there is a chance that they will have a higher net return. A classic example of these types of firms are start-ups, for which the success of their product or service is highly uncertain but which could result in incredibly high returns. These firms are willing to take on a very high chance of bankruptcy in order to try to "hit a homerun."

The line chart below helps represent the level of uncertainty that an individual or firm is willing to accept based on their level of risk aversion.



In general, the relationship between utilities with different risk aversion properties are as follows:

$$U(\text{Good outcome})_{\text{risk loving}} > U(\text{Good outcome})_{\text{risk averse}} > U(\text{Good outcome})_{\text{risk neutral}}$$

$$U(\text{Bad outcome})_{\text{risk loving}} < U(\text{Bad outcome})_{\text{risk neutral}} < U(\text{Bad outcome})_{\text{risk loving}}$$

Checking for Risk Aversion

Given a set of different utility functions, you can determine whether an individual or firm is risk averse, risk loving, or risk neutral by taking the derivative of the utility function with respect to the variable of interest (e.g., net return) and determining how a marginal change in the variable of interest affects utility.

- For a risk neutral utility, the first derivative of the utility function will be constant.
- For a risk averse utility, the first derivative of the utility function will indicate that for each additional one unit that your variable of interest increases, utility increases by less than one unit.
- For a risk loving utility, the first derivative of the utility function will indicate that for each additional one unit that your variable of interest increases, utility increases by more than one unit.

Example 3.1.7 Comparing Risk Aversion

Consider three individuals with three different utility functions of income, I.

Risk averse: $U_{ra} = \ln(I+5)$

Risk neutral: $U_{rn} = I+5$

Risk loving: $U_{rl} = \exp(I)$

Are the individuals' risk preferences correctly represented by the utility functions?

To determine this, we need to take the first order conditions of each utility function with respect to income, I.

$$\frac{dU_{ra}}{dI} = \frac{1}{(I+5)}$$

$$\frac{dU_{rn}}{dI} = 1$$

$$\frac{dU_{rl}}{dI} = \exp(I)$$

In the case of the risk neutral individual, the derivative shows that for each unit increase in income, utility will increase by a constant one unit.

In the case of the risk averse individual, increasing income by 1 unit would lead to a utility increase of 0.167 units. For an income increase of 2 units, utility will increase by 0.142 units. This clearly indicates that additional units of income will lead to a decreasing marginal increases in utility.

Lastly, in the case of the risk loving individual, increasing income by 1 unit would lead to a utility increase of 2.718 units. For an increase of 2 income units, utility will increase by 7.389 units. Clearly, this indicates that marginal increases of income will lead to an increasing marginal increases in utility.

3.1.3 Expected Utility

Empirical economics research has shown that most individuals and firms are *not* risk neutral. As such, it is necessary to capture the risk aversion preferences when calculating expected values.

We can do this by modeling different *utility functions*, which can reflect different levels of risk aversion. Then, we can calculate the *expected utility* rather than the expected value. The expected utility will provide a sense of how individuals and firms with different risk aversion levels incorporate uncertainty into their expected outcome calculation.

The expected utility can be calculated as

$$E[U(V)] = Prob(\text{Outcome}) \times (\text{Utility from outcome})$$

Consider again the weather forecast example from above: tomorrow's forecast calls for a 50% chance of rain that will result in 1 inch of precipitation. But now suppose that your utility from 1 inch of rain is 20 units. Then your expected utility is

$$E[U(\text{Precip})] = (0.50) \times U(1 \text{ in}) = 0.50 \times 20 = 10 \text{ units}$$

Example 3.1.8 Calculating Expected Utility

Let's re-examine the food cart example above. Suppose that you are again evaluating your potential income working at a food cart near a baseball stadium. Since you have to purchase your food cart permit ahead of time, you don't know what the conditions of the market will be at the time of a baseball game. For example, if the baseball team is doing well, the attendance might be much higher resulting in a higher return. Alternatively, if there is poor weather or the team is in last place, attendance might be very low and you might end up losing money if you are not able to offset the costs for your food cart permit.

Assume that the probability of a well-attended game is still 30% and that, in this situation, your net return would be \$100. The probability of a poorly attended game is 70% and your resulting net return would be –\$20.

Also assume that your utility from these outcomes is characterized by the utility function: $U(NR) = \frac{NR}{2}$. Your expected utility would be:

$$
\begin{aligned}
E[U(NR)] &= Prob(\text{Good attendance}) \times U(\text{NR if good attendance}) + \\
&= Prob(\text{Poor attendance}) \times U(\text{NR if poor attendance})
\end{aligned}
$$

$$
\begin{aligned}
E[U(NR)] &= (0.30) \times \left(\frac{\$100}{2}\right) + (0.70) \times \left(\frac{-\$20}{2}\right) \\
&= 8
\end{aligned}
$$

The expected utility of net returns is a positive 8 units. From a managerial decision standpoint, the positive expected utility suggests that you should purchase the permit and sell food from your cart near the stadium.

Example 3.1.9 Calculating Expected Utility with Different Risk Aversions

Suppose you're evaluating which of two firms is going to make an investment into the development of a product. The new product can lead to three net return outcomes: 25% chance of a net return of 20, 35% chance of a net return of 5, and a 40% chance of a net return of –10.

Firm A is risk averse and has the utility function: $U_A = \ln(NR + 20)$.

Firm B is risk neutral and has the utility function: $U_B = NR + 20$.

Compare and discuss the expected utilities of these two firms.

$$
\begin{aligned}
E[U(NR)_A] &= (0.25) \times (\ln(20+20)) + (0.35) \times (\ln(5+20)) + (0.40) \times (\ln(-10+20)) \\
&= 2.83
\end{aligned}
$$

$$
\begin{aligned}
E[U(NR)_B] &= (0.25) \times ((20+20)) + (0.35) \times ((5+20)) + (0.40) \times ((-10+20)) \\
&= 4.25
\end{aligned}
$$

The expected utility for both firms is positive, so there is evidence that both firms could invest in the new product development. However, it is obvious that the risk averse firm weighs more heavily the potential negative net return than the risk neutral firm.

How Does Uncertainty Relate to Statistical Analysis?

The big takeaway from this section is that you need at least two items for beginning to understand and manage uncertainty:

1. A model that describes outcomes under alternative economic conditions (e.g., profit function; income function).
2. Knowledge of the probability for each of the different outcomes.

In cases when risk aversion needs to be assessed or incorporated, you would also need to assume or determine a utility function that appropriately characterizes an individual or firm. We will mostly evaluate conditions under a risk neutrality assumption, but will at times come back to risk preferences in our analyses.

Typically, both items 1 and 2 are not immediately known. As such, we often have to refer and use historical data to estimate these outcomes and associated probabilities, as well as get a sense of how certain we are that our data provide reliable estimates. This is where statistical analysis comes into play.

3.2 Describing Random Variables

Uncertainty is the underlying concept of statistics. In social sciences, such as economics, almost everything that you analyze will have some level of uncertainty. This is because economics is the study of behaviors, which are always subject to uncertainty. This is the difference between theoretical economics and applied economics.

- ***Theoretical economics***: there is full certainty about outcomes. For example, if the demand function is $Q = 100 - 2P$, then at $P = 5$, we are fully certain that $Q = 90$.
- ***Applied economics***: there is uncertainty about outcomes. For example, a demand function is defined as $Q = 100 - 2P + e$, where e is an unknown, unexplainable factor (e.g., how

will people react if there is some highly unlikely or unprecedented weather or geopolitical event?). For example, at $P = 5$, the quantity demanded could be $Q = 90$, or $Q = 87$, or $Q = 91$.

Random variables: the uncertainty in outcomes is the reason that we use statistical tools. Any variable for which outcomes are uncertain until they actually occur are known as a *random variable*. For example, agricultural yields is a random variable because we do not know whether yields will be high, low, or average until the harvest occurs and we determine the yields. Another example of a random variable is stock returns. Until the end of a trading day, it is not certain whether stock returns will be higher, lower, or unchanged from the previous day.

If random variables are those for which outcomes cannot be determined with certainty until an outcome actually occurs, then is there a method for having some determining the likelihood with which an outcome will occur? For example, consider a fair, six-sided die. Before we actually roll the die, the outcome is unknown—it could be 1, or 2, or 6. However, even though we do not know the actual outcome, we do know that each of the six outcomes has an equal chance of rolling. That is, there is a one-in-six chance that a 1 will roll, that a 2 will roll, etc. The following table characterizes this concept.

Outcome	**Prob[Outcome]**
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Consider another example for which the probability of outcomes is not the same across outcomes. A baseball player usually has four opportunities to hit. That implies that the player can get anywhere from 0 to 4 hits. Suppose that the following table characterizes each of the outcomes and the probability of those outcomes.

# Hits	**Prob[# Hits]**
0	0.84
1	0.10
2	0.03
3	0.02
4	0.01

The table shows that there is a significantly higher probability of a hitter getting no hits (84% chance of zero hits) and the lowest probability is getting four hits (1% probability).

Probability density function (pdf) describes the relationship between an outcome and the probability of that outcome for a random variable. The pdf can be provided as either a function, a table, or a graph. The sum of all of the probabilities always equals one, because the likelihood of at least one of the possible outcomes occurring is 100%.

Probability density functions describe both ***discrete*** and ***continuous*** random variables. A discrete random variable is one where the outcomes can be described as specific values with no possible outcomes in between. For example, the number stocks in a portfolio would be 0, 1, 2, ... However, there could not be 3.332 number of stocks in the portfolio. Continuous random variables are those for which outcomes can take any value. For example, air temperature can be specified as a value as small as possible. Air temperature could be specified as 68 degrees Fahrenheit, or 68.4 degrees, or 68.42 degrees, or 68.424 degrees.

3.2.1 Graphical Characterization of a pdf

As any other function, the pdf describes the relationship between an input and the unique output. For example, for any value of x, the probability that x will occur is $f(x)$. Moreover, like any other function, the pdf can be graphically represented. Figure 3.1 provides a visual characterization of a pdf and how to use the pdf to determine the probability of an outcome. As the figure shows, the probability of outcome x_1 ($f(x_1)$) is higher than the probability of outcome x_2 ($f(x_2)$). Therefore, the likelihood that x_1 will occur is higher than the occurrence of x_2.

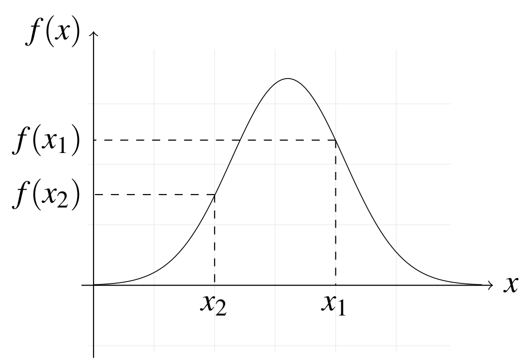


Figure 3.1: Visually Interpreting Probability Density Functions

3.2.2 Functional Characterization of a pdf

Once the functional form of a pdf is known, it is straightforward to determine the probability of any possibly outcome. For example, consider the binomial pdf, which determines the probability that k number of events will occur given n number of opportunities. One example of how this can be applied is to the baseball example described above. That is, the probability of k number of hits given n number of opportunities to hit. Suppose that the probability of getting at least one hit is $p = 0.25$. Then, what is probability that a player gets $k = 2$ hits provided that the player is given $n = 4$ opportunities.

The binomial pdf is defined as

$$f(k;n,p) = \left(\frac{n!}{k!(n-k)!}\right) p^k (1-p)^{n-k}$$

Therefore, we wish to solve

$$\begin{aligned} f(2;4,0.25) &= \left(\frac{4!}{2!(4-2)!}\right) 0.25^2 (1-0.25)^{4-2} \\ &= 0.21 \end{aligned}$$

That is, there is a 21% probability that a hitter gets 2 hits given 4 opportunities to hit.

3.2.3 Descriptive Statistics

Probability density functions are the Rolls Royces of the random variable world. They have all the possible bells and whistles one could desire. However, these abundant features can be overwhelming, making it difficult to characterize all of the car's numerous details. Most of the time, you want an efficient but effective vehicle such as a Toyota Camry, a middle-of-the-road model that provides a good representation of a car that is typically seen being driven on the road.

The pdf is extremely useful, because it fully characterizes the entire range of outcomes and the associated probabilities of those outcomes for a random variable. In most applications, however, knowing the entire range may be just too much information, which can dilute the amount and quality of inferences that can be drawn. A more manageable and more easily interpretable method for describing random variables would be by condensing the information contained in a pdf into a few generalizable measures. These measures are known as ***descriptive statistics*** of a random variable.

Typically, there are four measures (also referred to as the four moments) of a pdf that are used to describe that probability density function.

- ***Central tendency***: the central tendency of a probability density function describes the outcome that has the highest likelihood of occurrence. That is, without knowing with certainty the outcome of a random variable, the central tendency represents our best guess of the next outcome. Figure 3.2 provides a visual representation of the central tendency. That is, for a random variable x, the outcome μ_x has the highest likelihood of occurrence, $f(\mu_x)$, and, therefore, represents the central tendency of the pdf.

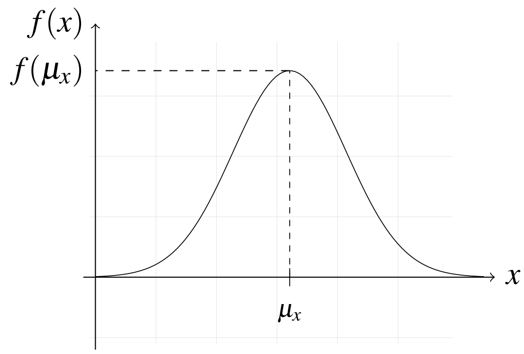


Figure 3.2: Central Tendency (Expected Value) of a Random Variable

There are a number of synonyms for central tendency. These include ***expected value***, ***mean***, ***average***, or the ***first moment of the pdf***. There are also different notations for the central tendency. For a random variable x, the central tendency of a population can be characterized as $E[x]$ or μ_x and of a sample as $\bar{x}$.

Three measures are typically used to described the central tendency of a pdf: the ***mean***, ***median***, and ***mode***.

  - Mean: the mean is the weighted sum of all of outcomes for a random variable, where the weights are the probabilities of occurrence for each outcome. That is, for the random variable x,

$$E[x] = \sum_{i=1}^{n} x_i \cdot f(x_i)$$

For example, recall the pdf for the number of hits that a baseball player gets during a game.

# Hits	Prob[# Hits]
0	0.84
1	0.10
2	0.03
3	0.02
4	0.01

The expected value of hits is,

$$\begin{aligned} E[\text{Hits} &= \textstyle\sum_{i=1}^{5} \text{Hits}_i \cdot \text{Prob}[\text{ Hits}] \\ &= (0 \cdot 0.84) + (1 \cdot 0.10) + (2 \cdot 0.03) + (3 \cdot 0.02) + (4 \cdot 0.01) \\ &= 0.26 \end{aligned}$$

Therefore, the most likely outcome is 0.26 hits. For example, in 100 at bats, the most likely result for the player is that he will have 26 hits.

– Median: the median is the value of the random variable such that half of all outcomes are less than that value and half of all outcomes is greater than that value. In the baseball hits example, the median is 2 hits, because there are two outcomes (0 and 1) that are less than 2 and two outcomes (3 and 4) that are greater than 2.

– Mode: the mode is the value of the random variable that occurs most frequently. This is the least used central tendency measure.

- ***Dispersion***: the dispersion describes the uncertainty of outcomes around the central tendency. It is a measure that provides information about the relative likelihood that an outcome of a random variable will be close in value to the central tendency.

  For example, suppose that for every 100 at bats, there are two baseball players, A and B, for whom the most likely outcome is that each gets 26 hits. Now, suppose that we consider five instances for which each player has 100 at bats. For baseball player A, the number of hits for each of the five 100 at bat opportunities are (29, 21, 22, 27, 26). For baseball play B, the outcomes are (45, 3, 14, 29, 11). If we were asked to place a bet about which player is more likely to be closest to the central tendency of 26 hits if each player had another 100 at bats, which player, A or B, would you choose? Most likely, you will place the bet on player A. That is, there is a higher risk of being wrong if the bet is placed on player B, because this player's hit performances are much more spread out around the central tendency than for player A.

  Figure 3.3 provides a visual representation of pdfs that have the same central tendency but different dispersion measures. The wider pdf represents a more risky (has greater dispersion) random variable. Conversely, the slimmer pdf is less risky (lower dispersion) because outcomes are more concentrated around the central tendency.

  There are a number of synonyms for dispersion that we will use interchangeably. These include ***variance***, ***spread***, or the ***second moment of the pdf***. There are also different notations for dispersion. For a random variable x, the dispersion of a population can be characterized as $Var[x]$ or σ_x^2 and of a sample as s_x^2.

  We will consider three measures that describe the dispersion of a pdf: the ***variance***, ***standard deviation***, and ***coefficient of variation***.

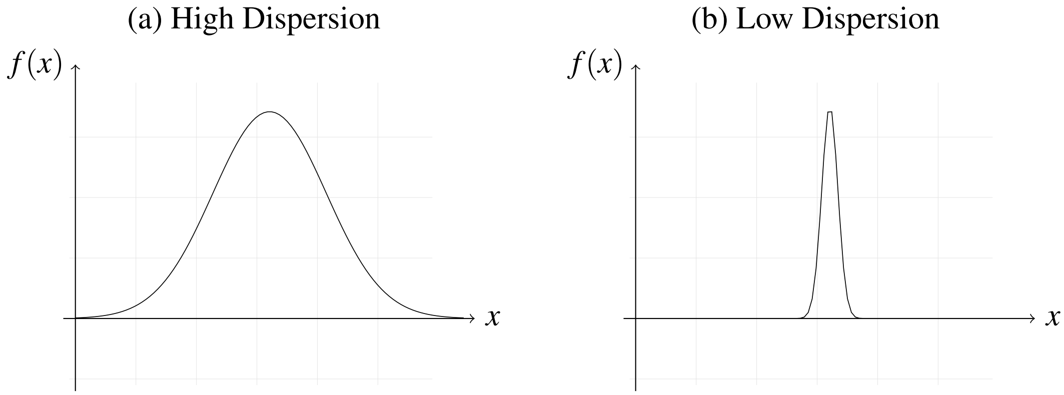


Figure 3.3: Differences in the Dispersion of a Random Variable

– Variance: the variance is the weighted sum of squared deviation of outcomes from the central tendency, where the weights are the probabilities of occurrence for each outcome. That is, for the random variable x,

$$\sigma_x^2 = \sum_{i=1}^{n} (x_i - E[x])^2 \cdot f(x_i)$$

For example, recall the pdf for the number of hits that a baseball player gets during a game.

# Hits	**Prob[# Hits]**
0	0.84
1	0.10
2	0.03
3	0.02
4	0.01

The expected value of hits is,

$$\begin{aligned} \sigma^2_{\text{Hits}} &= \textstyle\sum_{i=1}^{5} (\text{Hits}_i - E[\text{Hits}])^2 \cdot \text{Prob}[\text{Hits}] \\ &= (0-0.26)^2 \cdot 0.84 + (1-0.26)^2 \cdot 0.10 + (2-0.26)^2 \cdot 0.03 + \\ &\quad (3-0.26)^2 \cdot 0.02 + (4-0.26)^2 \cdot 0.01 \\ &= 0.49 \end{aligned}$$

The variance is 0.49 hits^2.

– Standard deviation: the deviation is a similar measure to the variance, except that it has the advantage of representing the dispersion in the units of the random variable.

That is, variance provides the dispersion in squared units. The standard deviation provides a more intuitive measure by presenting dispersion in the original units.

The standard deviation is calculated in the same manner for both populations and samples: it is the square root of the variance. For populations, the notation is $\sigma_x = \sqrt{\sigma_x^2}$ and for samples it is $s = \sqrt{s^2}$.

In the baseball hits example, the standard deviation is $\sigma_{\text{Hits}} = \sqrt{0.49} = 0.70$ hits.

- <u>Coefficient of variation</u>: the coefficient of variation (CV) is a unitless measure of dispersion. While the standard deviation represents dispersion in the original units (which is useful), we are still unable to compare dispersion measures across different random variables. By making the dispersion measure unitless, the CV—like an elasticity—allows us to compare relative dispersion/risk/uncertainty across different random variables.

  The coefficient of variation is the ratio of the standard deviation to the expected value. For a population, $CV_x = \dfrac{\sigma_x}{\mu_x}$ and for a sample, $CV_x = \dfrac{s_x}{\bar{x}}$.

  In the baseball hits example, the coefficient of variation is $CV_{\text{Hits}} = \dfrac{0.70}{0.26} = 2.69$.

- ***Skewness***: the skewness represents the degree to which the probability density function is asymmetric. That is, the degree to which the pdf is unequally distributed around the central tendency. A non-skewed (symmetric) pdf is one in which the mean is equal to the median and the mode. In a positively skewed pdf, the mean is greater than the median; that is, there are relatively more large valued outcomes, but these outcomes occur with a small probability (e.g., the pdf of prices is often described as being positively skewed). In a negatively skewed pdf, the mean is less than the median; that is, there are relatively more small valued outcomes, but these outcomes occur with a small probability (e.g., the pdf of agricultural crop yields is often described as being negatively skewed).

  Figure 3.4 provides a visual representation of skewness. The figure shows a symmetric (non-skewed) pdf, a positively skewed pdf, and a negatively skewed pdf.

- ***Kurtosis***: the kurtosis describes the peakness of a pdf. That is, it characterizes whether the peak of a probability density function is very high or not.

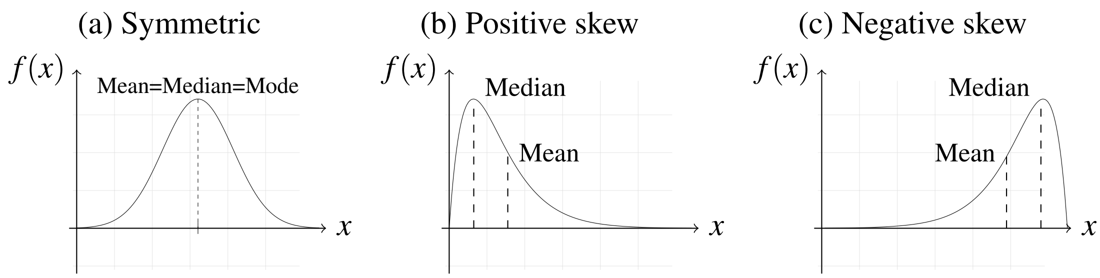


Figure 3.4: Characterization of Skewed Probability Density Functions

3.2.4 Hypothesis testing

Recall that statistical tools cannot be used to conclude anything with certainty. However, these tools can be used to evaluate whether there is enough empirical evidence (i.e., data-driven information) to support whether a particular outcome is observed more frequently than simply by chance.

As managers, you will be tasked to make decisions. However, you will also often need to provide (either for yourself or for your employer) support for your decisions, which will frequently require empirical evidence. Hypothesis tests provide a mechanism for using data to test questions that will help the decision-making process. In empirical economics, hypothesis tests are the foundation for developing data-based managerial decisions.

Structure of hypothesis testing

1. The foundation of a hypothesis test is similar to the U.S. judicial system—we always assume innocence. That is, we assume that some theoretical or previously established value is the initial assumption and we must initially believe that this is the true value. This is the ***null hypothesis***.

2. After the initial assumption of innocence is established, we specify the "guilty" condition. That is, we define the value that, if the theoretical or previously established value is incorrect, we believe to be the *actual* true value. This is the ***alternative hypothesis***.

3. Once the two conditions—theoretical and proposed—are specified, we must determine a value that can be used to test whether there is enough statistical evidence to support the new, proposed condition instead of the theoretical or previously established condition. This condition incorporates two sets of information: the distance/difference between the proposed and theoretical values and the level of uncertainty that we have that this difference occurred due to more than just chance. This is the ***test statistic***.

4. While a test statistic provides a ratio of the difference between the proposed and theoretical values and the uncertainty about that difference, we need some measure to which we can compare this ratio to determine whether there is enough evidence to support or not support the null hypothesis. This comparison measure is determined using a theoretical distribution of ratios. This measure is the ***critical statistic***. The critical statistic is conditional on how confident we wish to be when making a conclusion about whether the null hypothesis can be rejected.

5. Lastly, we compare the test statistic to the critical statistic. If the test statistic exceeds the critical statistic, we assume that there is enough statistical evidence in the observed data to reject the null hypothesis in favor the alternative hypothesis. That is, we conclude that there is enough evidence beyond a reasonable doubt to reject the initial null hypothesis of "innocence" and assume that the outcome that was observed in the data did not differ from the theoretical value due only to random chance. Otherwise, we must conclude that there is not enough statistical evidence to reject the initial assumption.

Mechanism of hypothesis testing

1. Define a null hypothesis. The null hypothesis will always be an equality, H_0 : (Observed value – Theoretical value) $= 0$.

2. Define an alternative hypothesis. The alternative can be in one of three forms,

$$
\begin{aligned}
H_a &: \quad (\text{Observed value} - \text{Theoretical value}) \quad > \quad 0 \\
H_a &: \quad (\text{Observed value} - \text{Theoretical value}) \quad < \quad 0 \\
H_a &: \quad (\text{Observed value} - \text{Theoretical value}) \quad \neq \quad 0
\end{aligned}
$$

3. Calculate a test statistic, t_{stat}. This calculation will be different based on the test you are performing.

   - For testing an observed mean value against some known/theoretically-informed value, the t-statistic is calculated as:

$$t_{\text{stat}} = \left(\frac{\text{Obs mean} - \text{Known/theory}}{s/\sqrt{n}} \right)$$

   - For testing simple correlation statistics, ρ, the t-statistic is calculated as:

$$t_{\text{stat}} = \rho \left(\sqrt{\frac{n-2}{1-\rho^2}} \right)$$

   - For testing an estimated regression coefficient, $\hat{\beta}$, the t-statistic is calculated as:

$$t_{\text{stat}} = \frac{\hat{\beta}}{\text{SE}_{\hat{\beta}}}$$

4. Determine the critical value from a t distribution table. The t critical value will depend on three conditions: the number of tails (one-tailed if the alternative hypothesis tests either the greater or less than condition; two-tailed if the alternative hypothesis tests the inequality condition), the statistical significance level (e.g., $\alpha = 0.05$), and the degrees of freedom.

5. Compare the test statistic to the critical value. For each of the three alternative hypothesis conditions, reject the null hypothesis if the following condition holds,

$$
\begin{aligned}
\text{For } H_a &: \quad (\text{Observed value} - \text{Theoretical value}) \quad > \quad 0 \quad \text{reject } H_0 \text{ if } t_{\text{stat}} > t_{\text{crit}} \\
\text{For } H_a &: \quad (\text{Observed value} - \text{Theoretical value}) \quad < \quad 0 \quad \text{reject } H_0 \text{ if } |t_{\text{stat}}| > t_{\text{crit}} \\
\text{For } H_a &: \quad (\text{Observed value} - \text{Theoretical value}) \quad \neq \quad 0 \quad \text{reject } H_0 \text{ if } |t_{\text{stat}}| > t_{\text{crit}}
\end{aligned}
$$

Example 3.2.10 Testing fertilizer treatment effects

Suppose you are testing the effects of a fertilizer treatment. There are two types of fertilizer, and you apply it to 25 different plots and observe the crop yields. The long-run average of production without fertilizer is 20 bushels per acre.

- Treatment 1: $Y_{\text{observed}} = 60$, $n = 25$, $s = 5$

$$\begin{aligned} H_0 : (60 - 20) &= 0 \\ H_a : (60 - 20) &\neq 0 \\ t_{\text{stat}} &= \left(\frac{60 - 20}{5/\sqrt{25}} \right) = 40 \\ t_{\text{crit}(\text{2-tailed}, \alpha=0.05, df=25-1)} &= 2.064 \\ |t_{\text{stat}}| &> t_{\text{crit}} \end{aligned}$$

  Because $|t_{\text{stat}}| > t_{\text{crit}}$, we reject the null hypothesis in favor of the alternative. In treatment 1, there is enough statistical evidence to suggest that applying fertilizer resulted in higher yields.

- Treatment 2: $Y_{\text{observed}} = 22$, $n = 25$, $s = 5$

$$\begin{aligned} H_0 : (22 - 20) &= 0 \\ H_a : (22 - 20) &\neq 0 \\ t_{\text{stat}} &= \left(\frac{22 - 20}{5/\sqrt{25}} \right) = 2 \\ t_{\text{crit}(\text{2-tailed}, \alpha=0.05, df=25-1)} &= 2.064 \\ |t_{\text{stat}}| &< t_{\text{crit}} \end{aligned}$$

  Because $|t_{\text{stat}}| < t_{\text{crit}}$, we cannot reject the null hypothesis in favor of the alternative. In treatment 2, there is not enough statistical evidence to suggest that applying fertilizer resulted in higher yields.

3.2.5 p-values

P-values are another method for interpreting statistical outcomes an whether to reject the null hypothesis. If the null hypothesis was actually true, the "p-value" describes the probability that in another similar experiment you would observe the same outcome as the one that was observed using the current data. That is, it provides the likelihood that the null hypothesis is true.

If p-values are small, then we know that there is a high likelihood that the null hypothesis is not true. Therefore, for small p-values, we would reject the null hypothesis in favor of the alternative hypothesis.

Example 3.2.11 Using p-values for hypothesis testing

Consider two hypothesis tests in which you found the following p-values.

- p-value = 0.09. In this case, you would reject the null hypothesis with $(1-0.09) = 91\%$ confidence level.
- p-value = 0.54. in this case, you would reject the null hypothesis with $(1-0.54) = 46\%$ confidence level.

In economics, any p-value≤ 0.10 is considered sufficient statistical evidence to reject the null hypothesis in favor of the alternative hypothesis. For p-value> 0.10, there would not be enough statistical evidence to reject the null hypothesis.

3.3 Relationship Between Two Variables

Developing methods for understanding relationships between variables is an important tool for managerial economists. We will consider three approaches—covariance, correlation, and regression—for analyzing the relationships. While it will be important to recognize and become proficient in applying these methods, it is perhaps more important to understand why developing these relationships using actual, real-world data is helpful for making optimal decisions.

The relationship between variables can offer important insights about two questions: (i) Will there be a change resulting from a managerial decision? and (ii) What is the magnitude of this change? This is precisely the connection between the theoretical concepts and real-world data that we are striving to make.

Throughout this section, we will use the following sample data of the number of sales that are made by seven firms and the number of television advertisements that the firm purchased.

Y=Sales	X=TV Ads
120	10
95	5
110	12
125	15
80	9
100	10
150	15

3.3.1 Covariance

The covariance describes the comovement of two variables. That is, covariance is a measure of whether the values of two variables change in similar or opposite directions.

$$\text{Cov} = S_{yx} = \sum_{i=1}^{n} \frac{(y_i - \bar{y})(x_i - \bar{x})}{n-1}$$

Mechanism

1. Calculate the mean of each variable.

$$\begin{aligned} \bar{y} &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n} y_i = 111.4 \\ \bar{x} &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n} x_i = 10.9 \end{aligned}$$

2. Calculate the covariance.

$$S_{yx} = \frac{[(10-10.9)(120-111.4)+(5-10.9)(95-111.4)+(12-10.9)(110-111.4)\ldots]}{7-1} = 61.9$$

If $S_{yx} > 0$, then deviations from the mean of y occur in the same direction as deviations from the mean of x. For example, more sales than the average number of sales are observed when there are more advertisements than the average number of advertisements. Similarly, fewer sales than the average number of sales are observed when there are fewer advertisements than the average number of advertisements.

If $S_{yx} < 0$, then deviations from the mean of y occur in the opposite direction as deviations from the mean of x. For example, more sales than the average number of sales are observed when there are fewer advertisements than the average number of advertisements. Similarly, fewer sales than the average number of sales are observed when there are more advertisements than the average number of advertisements.

While covariance provides insights about the direction of comovements in two variables, it does not provide a sense of the strength of the comovement relationship.

3.3.2 Correlation

Correlations offer a more comprehensive measure of variable comovement. It is a unit-less measure that has a range of $[-1, 1]$.

$$\rho_{yx} = \frac{S_{yx}}{S_y S_x}$$

<u>Mechanism</u>

1. Calculate the covariance, S_{yx}.
2. Calculate the standard deviations for y and x.
3. Determine the correlation statistic, ρ_{yx}.

For the sample data, $S_{yx} = 61.9$, $S_y = 22.86$, and $S_x = 3.53$.

$$\rho_{yx} = \frac{61.9}{22.86 \cdot 3.53}$$

The correlation coefficient provides information about both the direction and strength of the comovement relationship. The direction is interpreted similarly to covariance. The strength of the relationship is determined by the closeness of the correlation coefficient to the bounds 1 or -1. The closer the correlation coefficient is to a bound, the strong the relationship. The closer the correlation coefficient is to zero, the weaker the relationship.

While the magnitude of the correlation coefficient is important, it is necessary to test whether there is enough statistical significance to suggest that the magnitude is greater or less than zero. That is, whether there is a significant relationship among two variables. We can do so by setting up a hypothesis test.

$$\begin{aligned}
H_0 : \rho_{yx} &= 0 \\
H_a : \rho_{yx} &\neq 0 \\
t_{\text{stat}} &= \rho \sqrt{\tfrac{n-2}{1-\rho^2}} = 2.67 \\
t_{\text{crit}(2\text{-tail},\alpha=0.05,df=n-2)} &= 2.53
\end{aligned}$$

In the sales and TV advertisement example, $|t_{\text{stat}}| > t_{\text{crit}}$, implying that we have enough statistical evidence to reject the null hypothesis in favor of the alternative hypothesis.

The correlation provides a useful measure of the comovement of two variables, but it does not provide a means to specify a functional form of how the variables are related.

3.3.3 Regression

Recall that we have dealt with demand functions of numerous functional forms. For example, we have considered a general linear demand function, $Q = \alpha - \beta P$ and then used information about elasticity measures to derive the unknown parameters α and β. However, many times elasticity information may be difficult to obtain. What is an alternative approach to determining the unknown parameters?

Suppose that we had information about what quantity demanded we observed at a certain price. If we had enough of this type of information, we could determine the demand schedule (i.e., the demand curve). A regression analysis provides a methodology to estimate the demand schedule from information that is observed either in the market or your firm.

But, recall that when we deal with real-world data, there is always some degree of uncertainty that we must consider. In an estimated demand function, this is represented by an error term, e. That is,

$$Q = \alpha - \beta P + e$$

Intuitively, the term e represents the information about individuals' demand for a product that cannot be explained by prices. That is, it is other factors that influence individuals' decisions to purchase or not purchase a good *other* than price.

The objective of a regression analysis is to identify the unknown parameters α and β while at the same time minimizing the amount of error, e.

Identifying the Regression Parameters: the parameters of a linear model are identified subject to the constraint that amount of error, e, is minimized. That is, consider that for each observation in a dataset, we specify a general model, $y_i = \alpha + \beta x_i + e_i$. This implies that $e_i = y_i - \alpha - \beta x_i$.

The objective is to minimize the sum of squared errors across all of the observations in the dataset.

$$\min \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$$

To minimize the sum of squared errors with respect to α and β, we take the first-order conditions with respect to each of the parameters.

$$\begin{aligned}
\frac{\partial \sum_{i=1}^{n} e_i^2}{\partial \alpha} &= \sum_{i=1}^{n} -2(y_i - \alpha - \beta x_i) = 0 \\
&= \sum_{i=1}^{n} (y_i - \alpha - \beta x_i) = 0 \\
&= \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \alpha - \beta \sum_{i=1}^{n} x_i = 0 \\
&= \sum_{i=1}^{n} y_i - n\alpha - \beta \sum_{i=1}^{n} x_i = 0 \\
&= \frac{1}{n} (\sum_{i=1}^{n} y_i - n\alpha - \beta \sum_{i=1}^{n} x_i) = 0 \\
&= \frac{1}{n} \sum_{i=1}^{n} y_i - \alpha - \beta \frac{1}{n} \sum_{i=1}^{n} x_i = 0
\end{aligned}$$

Solving for α results in the formula for determining the estimated coefficient, $\hat{\alpha}$.

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$

This indicates that to determine the value of the parameter α, we need the mean values of y and x and the estimated value of the β coefficient. To solve for the estimated value of β, we need to take the first-order conditions of $\sum_{i=1}^{n} e_i^2$ with respect to β.

$$\begin{aligned}
\frac{\partial \sum_{i=1}^{n} e_i^2}{\partial \beta} &= \sum_{i=1}^{n} -2x_i(y_i - \alpha - \beta x_i) = 0 \\
&= \sum_{i=1}^{n} y_i x_i - \alpha \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_i^2 = 0 \\
&= \frac{1}{n} \sum_{i=1}^{n} y_i x_i - \alpha \frac{1}{n} \sum_{i=1}^{n} x_i - \beta \frac{1}{n} \sum_{i=1}^{n} x_i^2 = 0 \\
&= \frac{1}{n} \sum_{i=1}^{n} y_i x_i - (\bar{y} - \beta\bar{x}) \frac{1}{n} \sum_{i=1}^{n} x_i - \beta \frac{1}{n} \sum_{i=1}^{n} x_i^2 = 0 \\
&= \frac{1}{n} \sum_{i=1}^{n} y_i x_i - (\bar{y} - \beta\bar{x})\bar{x} - \beta \frac{1}{n} \sum_{i=1}^{n} x_i^2 = 0 \\
&= \frac{1}{n} \sum_{i=1}^{n} y_i x_i - \overline{yx} - \beta (\frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2) = 0
\end{aligned}$$

Solving for β will provide the estimate for the coefficient.

$$\hat{\beta} = \frac{\frac{1}{n}\sum y_i x_i - \overline{yx}}{\frac{1}{n}\sum x_i^2 - \bar{x}^2} = \frac{\frac{1}{n}\sum (y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{n}\sum x_i^2 - \bar{x}^2} = \frac{S_{yx}}{S_x^2}$$

In the sales and TV advertisement example, $\bar{y} = 111.4$, $\bar{x} = 10.9$, $S_{yx} = 61.9$, $S_x^2 = 10.7$. The estimated coefficients are

$$\begin{aligned}
\hat{\beta} &= \frac{61.9}{10.7} = 5.79 \\
\hat{\alpha} &= 111.4 - (5.79)(10.9) = 48.3
\end{aligned}$$

Therefore, the estimated sales equation can be written as follows

$$\text{Sales} = 48.3 + 5.79\text{Ads}$$

The interpretation of the estimated coefficients must be done with caution. That is, these must be interpreted as correlations.

3.3.4 Post-estimation Analysis

The uncertainty associated with regression analysis (or any statistical analysis) requires that we determine whether there is enough statistical evidence to support that the estimated parameters are different from zero. In a regression analysis, the test statistic incorporates information about the magnitude of the estimated coefficient and the uncertainty of that estimate.

To determine the uncertainty about the estimates, we need to consider how well (or poorly) the parameters do in predicting an outcome that was actually observed. That is, we need to consider the size of the error, e, and more specifically, the value that we attempted to minimize: the sum of squared errors, $S_e^2 = \sum e_i^2$.

Mechanism

1. Using the estimated coefficients/function, determine the predicted values for each observation in a dataset. That is, calculate $\hat{y}_i = \hat{\alpha} + \hat{\beta}x_i$.
2. Determine the estimate error, $\hat{e}_i$, for each observation by taking the difference between an actual observation, y_i, and the predicted value, $\hat{y}_i$. That is, calculate $\hat{e}_i = y_i - \hat{y}_i$.
3. Square each of the errors for each observation, $\hat{e}_i^2$.
4. Add up all of the estimated squared errors. That is, calculate the Sum of Squared Errors: SSE.

Example 3.3.12 Calculating the Sum of Squared Errors

Y	X	$\hat{Y}$	$\hat{e}$	$\hat{e}^2$
120	10	$48.29 + 5.79(10) = 106.19$	$120 - 106.19 = 13.81$	$13.81^2 = 190.72$
95	5	$48.29 + 5.79(5) = 77.24$	$95 - 77.24 = 17.76$	$17.76^2 = 315.42$
110	12	$48.29 + 5.79(12) = 117.77$	$110 - 117.77 = -7.77$	$-7.77^2 = 60.37$
125	15	$48.29 + 5.79(15) = 135.14$	$125 - 135.14 = -10.14$	$-10.15^2 = 102.82$
80	9	$48.29 + 5.79(9) = 100.40$	$80 - 100.40 = -20.40$	$-20.40^2 = 416.16$
100	10	$48.29 + 5.79(10) = 106.19$	$100 - 106.19 = -6.19$	$-6.19^2 = 38.32$
150	15	$48.29 + 5.79(15) = 135.14$	$150 - 135.14 = 14.86$	$14.86^2 = 220.82$

The sum of all $\hat{e}^2$ values is $SEE = \sum_{i=1}^{7} \hat{e}^2 = 1344.62$.

The sum of squared errors value allows us to perform a number of post-estimation analyses. We will focus on two: determining the statistical significance of estimated parameter values and determining the overall fit of the regression model.

Standard Error Estimation
Standard errors provide an opportunity to evaluate the uncertainty around an estimated parameter value. In the regression model, $Y = \alpha + \beta X + e$, the standard error associated with the estimated parameter $\hat{\beta}$ is

$$\text{SE}_{\hat{\beta}} = \sqrt{\frac{SSE/(n-k)}{S_x^2}}$$

where n is the number of observations in the sample, k is the number of total parameters in the regression model, and $S_x^2 = \sum_{i=1}^{n}(x_i - \overline{x})^2$.

Hypothesis Testing

Once the standard error is estimated, we can perform hypothesis tests of the estimated parameter values. Typically, we will be interested in testing whether an estimated parameter value is statistically different from zero. That is, is there enough statistical evidence that the estimated relationship between the dependent variable Y and the independent variable X is not just due to a statistical anomaly.

The hypothesis test is as follows.

$$H_0: \quad \hat{\beta} - 0 = 0$$

$$H_a: \quad \hat{\beta} \neq 0$$

$$t_{\text{stat}}: \quad \frac{\hat{\beta}}{\text{SE}_{\hat{\beta}}}$$

$$t_{\text{crit}}: \quad t_{\text{crit (2-tailed},\alpha=0.05,df=n-k)}$$

Example 3.3.13 Hypothesis Test for an Estimated Parameter

$$H_0: \quad \hat{\beta} - 0 = 0$$

$$H_a: \quad \hat{\beta} \neq 0$$

$$t_{\text{stat}}: \quad \frac{5.79}{2.45} = 2.36$$

$$t_{\text{crit}}: \quad t_{\text{crit (2-tailed},\alpha=0.05,df=5)} = 2.571$$

The evaluation criteria, $|t_{\text{stat}}| < t_{\text{crit}}$, indicates that we cannot reject the null hypothesis that $\beta = 0$.

Hypothesis Testing Rule of Thumb: If there is a sufficient number of observations, $n > 30$, then we can use a useful rule of thumb to quickly evaluate the statistical significance of estimated parameter values. That is, an estimated parameter is statistically significant at the 95% confidence level if

$$2 \cdot \mathrm{SE}_{\hat{\beta}} < \hat{\beta}$$

Regression Goodness of Fit: The R-squared Measure

The sum of squared errors can also be used to determine the overall fit of the estimated regression. For linear regression models, he most common method for evaluating the model fit is by using the R-squared statistic. The R-squared explains the amount of information in the dependent variable that can be explained by information in the regressors.

There are two relevant measures for calculating the R-squared statistic: the sum of squared errors, SSE, and the sum of squared deviations from the mean of the dependent variable, $S_y^2 = \sum(Y_i - \overline{Y})^2$. The first measure, SSE, represents the proportion of information in the dependent variable that *cannot* be explained by the regressors. The second term, S_y^2, represents the proportion of information in the dependent variable that can be explained by simply using the average of the dependent variable (i.e., using a crude measure).

The R-squared measures is calculated as follows

$$R^2 = 1 - \frac{SSE}{S_y^2}$$

Intuitively, the R-squared represents the proportion of information that *was* explained by using the regression analysis. Another way to interpret the R-squared is that it is the squared correlation between the dependent and independent variables. Therefore, the R-squared measure is bounded between 0 and 1. An R-squared measure that is closer 1 implies that the explanatory power of the regression model is stronger. As R-squared nears 0, the explanatory power is weaker.

Adjusted R-squared: The R-squared measure defined above is known as the "unadjusted" R-squared. It is unadjusted because, by definition, as we add additional variables into the regression equation, the R-squared increases. However, what if adding the extra variable only adds complexity to the model (remember, as economists we want the simplest, most efficient model) without actually improving our ability to explain information in the dependent variable? If that's the case, then adding the extra variable is *not* optimal.

To weigh the trade-offs between increasing model complexity (i.e., adding a new variable) and how much useful information is added by the extra variable, we can calculate an *adjusted* R-squared. The adjusted R-squared directly accounts for this trade off in the calculation of the R-squared measure.

$$\text{adj } R^2 = 1 - \frac{SSE \times (n-1)}{S_y^2 \times (n-k)}$$

The adjusted R-squared is a slightly modified version of the unadjusted R-squared. The adjustment is the ratio between essentially the number of observations, $(n-1)$, and the degrees of freedom, $(n-k)$. If we add useless variables, the SSE measure in the numerator will remain relatively the same while the $(n-k)$ measure in the denominator will increase. As a result, the adjusted R-squared measure will decrease, indicating that we increased the model complexity without improving our ability to explain information in the dependent variable. If, however, the adjusted R-squared measure increases, then we know that increasing the model complexity was worthwhile because we are able to better explain Y.

Example 3.3.14 Calculating the R-squared Measures

In the sales and TV advertisements example, we know the following

$$\begin{aligned} n &= 7 \\ k &= 2 \\ SSE &= 1344.62 \\ S_y^2 &= 3135.71 \end{aligned}$$

The unadjusted and adjusted R-squared measures are

$$\begin{aligned} R^2 &= 1 - \left(\tfrac{1344.62}{3135.71}\right) = 0.571 \\ \text{adj } R^2 &= 1 - \left(\tfrac{1344.62}{3135.71}\right)\left(\tfrac{7-1}{7-2}\right) = 0.485 \end{aligned}$$

3.3.5 Forecasting

An important reason for estimating regression models is to perform post-estimation forecasting. That is, using the estimated regression model to predict future potential outcomes. However, it is also important to understand the degree of error in the forecast.

To forecast the value of a dependent variable, Y_f, for some potential value of the independent variable, X_f, calculate

$$Y_f = \hat{\alpha} + \hat{\beta} X_f$$

The measure of uncertainty around this forecast will be the 95% confidence interval.

Structure

1. Determine the sum of squared error of the forecast.
2. Use the forecast sum of squared error to calculate the forecast standard error.
3. Use the forecast standard error to estimate the 95% forecast confidence interval.

Mechanism

1. After determining the regression SSE, the adjusted forecast sum of squared errors ($fSSE$):

$$fSSE = SSE\left\{1+\frac{1}{n}+\frac{(x_f-\bar{x})^2}{nS_x^2}\right\}$$

2. Calculate $SE_f = \sqrt{fSSE}$.
3. Determine the 95% forecast confidence interval:

$$[\hat{Y}_f - 1.96SE_f, \hat{Y}_f + 1.96SE_f]$$

Example 3.3.15 Forecasting and Evaluating Forecast Uncertainty

For the sales and TV advertisements example, suppose that we wish to forecast the sales volume when we purchase 10 TV ads. That is,

$$\widehat{\text{Sales}}_f = 48.29 + 5.79(10) = 106$$

Our prediction is 106 units in sales. However, we know that there is likely uncertainty around this projected sales volume. We wish to determine the range of sales around this forecast value that captures 95% of potential sales outcomes.

1. Calculate $fSSE$

$$fSSE = 1344.62\left\{1+\frac{1}{7}+\frac{(10-10.9)^2}{7\cdot 12.46}\right\} = 1549.20$$

2. $SE_f = \sqrt{1549.20} = 39.36$
3. Determine the confidence interval

$$[106 - 1.96\cdot 39.36, 106 + 1.96\cdot 39.36] = [29, 183]$$

This implies that the forecasted sales volume is 106 units, but that when we purchase 10 TV ads, 95% of sales outcomes will be between 29 and 183 units.

3.4 Regression with Multiple Independent Variables

So far, we have assessed relationships among two variables. For example, we have considered demand models of the form, $Q_i = \alpha + \beta P_i + e$. While this is a good starting point for understanding the fundamentals of regression analysis, the assumption that purchasing behavior depends only on the price of the good or service is quite strong and not realistic for most goods and services. Therefore, it is necessary to consider how more complete demand models can be estimated using regression analysis. Specifically, how do we account and estimate the impacts of exogenous shifters on consumption behavior?

Consider a general model for the demand of good i,

$$Q_i = \alpha + \beta_1 P_i + \beta_2 P_j + \beta_3 I + \ldots + e$$

In this model, we may include the price of the good, the price of other goods, income, among other factors that we discussed Chapter 4. Our goal, then, is to use data about the consumption of good or service i and other factors to estimate the marginal effects of these multiple variables on consumption behavior.

Structure of Empirical Demand Analysis: when starting an empirical analysis to estimate demand functions, it is useful to follow a general set of rules.

1. Develop a theoretical model of the demand function for the good or service of interest. This should be founded in economic theory and market/industry information about factors that can affect the consumption of the good or service.
2. Collect as much data as possible about the variables in the theoretical demand model.
3. Generate standard summary statistics about the variables in the model.
4. Generate visual relationships (e.g., scatter plots) of all variable pairs.
5. Based on the theoretical economic model, the available data, and the preliminary data analysis in steps (3) and (4), define a regression model for the demand function.
6. Estimate the parameters in the regression model.
7. Evaluate the statistical and economic significance of the parameters.
8. Perform post-regression fit analyses.
9. Interpret the estimation results within an economic framework.

3.4.1 Estimating Parameters in a Multiple Regressor Demand Model

Consider the following demand model with two explanatory regressor variables.

$$Q_i = \alpha + \beta_1 P_i + \beta_2 I + e$$

where I represents income. One of the most critical aspects necessary for estimating a multiple regressor demand model is the variance-covariance matrix. This is a table that represents the relationships (variances and covariances) across the variables in the model. For the above model, the variance-covariance matrix is as follows.

	Q_i	P_i	I
Q_i	S_Q^2	$S_{Q,P}$	$S_{Q,I}$
P_i	$S_{Q,P}$	S_P^2	$S_{P,I}$
I	$S_{Q,I}$	$S_{P,I}$	S_I^2

The diagonal of this table represents the variances of each variable, S_Q^2, S_P^2, S_I^2. The values off of the diagonal are the covariances between the variables.

Using the variance-covariance matrix, it is then possible to generate a correlation table.

	Q_i	P_i	I
Q_i	$\rho_{Q,Q} = \frac{S_Q^2}{S_Q S_Q} = 1$	$\rho_{Q,P} = \frac{S_{Q,P}}{S_Q S_P}$	$\rho_{Q,I} = \frac{S_{Q,I}}{S_Q S_I}$
P_i	$\rho_{Q,P} = \frac{S_{Q,P}}{S_Q S_P}$	$\rho_{P,P} = \frac{S_P^2}{S_P S_P} = 1$	$\rho_{P,I} = \frac{S_{P,I}}{S_P S_I}$
I	$\rho_{Q,I} = \frac{S_{Q,I}}{S_Q S_I}$	$\rho_{P,I} = \frac{S_{P,I}}{S_P S_I}$	$\rho_{I,I} = \frac{S_I^2}{S_I S_I} = 1$

The values on the diagonal of this table are all equal to 1, because the correlation between a variable and itself is always one (the variables are perfectly correlated because they are the same).

With the variance-covariance and the correlation tables in hand, we have all of the necessary information to estimate the model parameters, α, β_1, and β_2.

$$\begin{aligned}
\hat{\beta}_1 &= \frac{S_Q}{S_P} \cdot \frac{\rho_{Q,P} - \rho_{Q,I}\rho_{P,I}}{1 - \rho_{P,I}^2} \\
\hat{\beta}_2 &= \frac{S_Q}{S_I} \cdot \frac{\rho_{Q,I} - \rho_{Q,P}\rho_{P,I}}{1 - \rho_{P,I}^2} \\
\hat{\alpha} &= \overline{Q} - \hat{\beta}_1 \overline{P} - \hat{\beta}_2 \overline{I}
\end{aligned}$$

Note that the parameter estimates now represent the need to consider the relationship between not only the dependent variable (quantity demanded) and each independent regressor variable, but also the relationship between the two regressors, price and income ($\rho_{P,I}$). If there is no statistical relationship, then $\rho_{P,I} = 0$ and the formulas for each parameter estimate simplify to those in the one-regressor estimation model.

3.4.2 Estimating Post-Regression Statistics

After the parameters of the demand model are estimated, it is necessary to determine the fit statistics in order to test the hypotheses that the estimates are statistically different from zero.

These require the calculation of the sum of squared errors and the sum of squared deviations from the dependent variable mean. That is,

$$
\begin{aligned}
SSE_{\hat{e}} &= \textstyle\sum_{i=1}^{n}(Q_i - \hat{Q}_i)^2 = \sum_{i=1}^{n}\hat{e}_i^2 \\
\tilde{S}_Q^2 &= \textstyle\sum_{i=1}^{n}(Q_i - \bar{Q}_i)^2
\end{aligned}
$$

Once these statistics are determined, you can calculate the standard errors, t statistics, and the adjusted R^2 statistic.

$$
\begin{aligned}
SE_{\hat{\beta}_1} &= \sqrt{\frac{SSE}{\tilde{S}_P^2(1-\rho_{P,I}^2)(n-k)}} \\
SE_{\hat{\beta}_2} &= \sqrt{\frac{SSE}{\tilde{S}_I^2(1-\rho_{P,I}^2)(n-k)}}
\end{aligned}
$$

$$
\begin{aligned}
t_{\hat{\beta}_1} &= \frac{\hat{\beta}_1}{SE_{\hat{\beta}_1}} \\
t_{\hat{\beta}_2} &= \frac{\hat{\beta}_2}{SE_{\hat{\beta}_2}}
\end{aligned}
$$

where k is the number of parameters that you estimate in the model, and $\tilde{S}_P^2 = \sum(P - \bar{P})^2$ and $S_I^2 = \sum(I - \bar{I})^2$.

3.4.3 Exponential Demand Functions

Recall that we have discussed numerous alternatives for describing demand functions. That is,

$$
\begin{aligned}
\text{Linear:} \quad Q &= \alpha - \beta P \\
\text{Quadratic:} \quad Q &= \alpha - \beta_1 P + \beta_2 P^2 \\
\text{Exponential:} \quad Q &= \alpha P^{\beta}
\end{aligned}
$$

For the linear and quadratic forms, we can use the regression estimation methods with which we are currently familiar. However, this is not the case with the exponential form of the demand function.

However, if we first linearize the exponential function, the standard regression approach would be appropriate. That is, consider the log-log transformed form of the exponential demand function, which is obtained by taking the logarithm of both sides of the exponential demand function.

Consider the transformation of the empirical exponential equation.

$$\begin{aligned} Q_i &= \alpha P_i^{\beta_1} P_j^{\beta_2} e \\ \ln Q_i &= \ln \alpha + \beta_1 \ln P_i + \beta_2 \ln P_j + \ln e \end{aligned}$$

If we let $\ln Q_i = \tilde{Q}_i$, $\ln \alpha = A$, $\ln P_i = \tilde{P}_i$, $\ln P_j = \tilde{P}_j$and $\ln e = v$, the log-transformed equation becomes,

$$\tilde{Q}_i = A + \beta_1 \tilde{P}_i + \beta_2 \tilde{P}_j + v$$

which can be estimated using the standard regression approach and testing the statistical significant of the parameters and the regression fit using the standard methods.

The useful aspect of using the log-log transformed demand model is that the estimated parameters can be directly interpreted as elasticities, which makes for more straightforward economic interpretation. However, the estimated model must be transformed back into its original exponential form to be interpreted as it was originally expressed. That is, the original exponential demand functional form can be obtained after estimating the linearized model as follows.

$$Q_i = e^{\hat{\alpha}} P_i^{\hat{\beta}_1} P_j^{\hat{\beta}_2}$$

Example 3.4.16 Converting Log-Log Models to Exponential Form

Suppose the estimated log-log demand model is,

$$\tilde{Q}_i = 1.45 - 2.15\tilde{P}_i + 0.64\tilde{P}_j$$

The original exponential form is,

$$\begin{aligned} Q_i &= e^{1.45} P_i^{-2.15} P_j^{0.64} \\ Q_i &= 4.26 P_i^{-2.15} P_j^{0.64} \end{aligned}$$

Chapter 4: Review of Economics Foundations

4.1 The Demand Function

The demand function represents the amount of a good or service that a person is willing to acquire at a certain price within some specified time period. A general characterization of a demand function is

$$Q_i^D = f(P_i | x_1, x_2, \ldots x_k)$$

where Q_i^D represents the quantity demanded for good i, P_i is the price, and $x1, \ldots x_k$ are other exogenous factors that can influence the demand for the good or service.

The *Law of Demand* states that for normal goods, increases in the price of good i will lead to a decrease in the quantity demanded of that good. That is,

$$\frac{dQ_i^D}{dP_i} \leq 0$$

When we specify a demand function, we assume that all of the other exogenous factors $(x_1, \ldots x_k)$ are held constant and do not change. That is, we assume *ceteris parabus* conditions. However, if one or more of these factors does change, the demand curve can shift up or down, rotate, or both. Examples of exogenous demand curve shifters are:

- Income
- Price of other complementary or substitute goods
- Consumer expectations about future conditions
- Size of a market
- Consumers' tastes and preferences
- Demographic and socioeconomic characteristics of consumers within a market

In this course, we will work with primarily three forms of the demand function: linear, exponential, and quadratic.

$$\begin{aligned}
\text{Linear form:} \quad & Q_i^D &=& \ \alpha - \beta P \\
\text{Exponential form:} \quad & Q_i^D &=& \ \alpha P^{\beta}, \quad \text{where } \beta < 0 \\
\text{Quadratic form:} \quad & Q_i^D &=& \ \alpha + \beta P - \gamma P^2, \quad \text{where } P \geq \beta/2\gamma
\end{aligned}$$

Figure 4.1 provide a visual representation of these three demand functional forms. As made evident by the figure, only the portions of the function that satisfy the *Law of Demand* are shown.

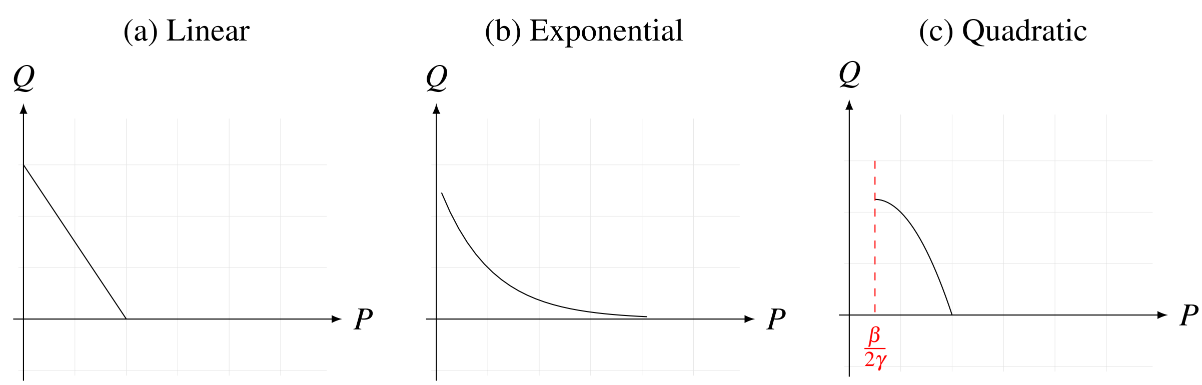


Figure 4.1: Three Functional Forms of the Demand Function

As shown in Figure 4.1, the plotted demand functions do not appear like the demand curve graphs that are typically presented in economics courses. That is, the quantity, Q, is on the y-axis and the price, P, is on the x-axis. The form that you have most likely encountered is the inverse demand. That is,

$$P_i = g(Q_i^D | x_1, x_2, \ldots x_k)$$

To determine the inverse demand, simply solve for price as a function of quantity. Conversely, to derive the demand form, solve for quantity as a function of price.

4.2 The Supply Function

The supply function represents the amount of a good or service that a firm is willing to sell at a certain price within some specified time period. A general characterization of a supply function is

$$Q_i^S = f(P_i | z_1, z_2, \dots z_k)$$

where Q_i^S represents the quantity supplied of good i, P_i is the price, and $z1, \dots z_k$ are other exogenous factors that can influence the supply of the good or service.

The *Law of Supply* states that for normal goods, increases in the price of good i will lead to an increase in the quantity supplied of that good. That is,

$$\frac{dQ_i^S}{dP_i} \geq 0$$

When we specify a supply function, we assume that all of the other exogenous factors $(z_1, \dots z_k)$ are held constant and do not change. That is, we assume *ceteris parabus* conditions. However, if one or more of these factors does change, the supply curve can shift up or down, rotate, or both. Examples of exogenous supply curve shifters are:

- Input prices
- Technology and production efficiency/effectiveness
- Size of the market/number of competing producers
- Firms' expectations about the future
- Production conditions (e.g., weather)

In this course, we will work with primarily three forms of the demand function: linear, exponential, and quadratic.

$$\begin{aligned}
\text{Linear form:} \quad & Q_i^S &=& \ \alpha + \beta P \\
\text{Exponential form:} \quad & Q_i^S &=& \ \alpha P^{\beta}, \quad \text{where } \beta > 0 \\
\text{Quadratic form:} \quad & Q_i^S &=& \ \alpha - \beta P + \gamma P^2
\end{aligned}$$

Figure 4.2 provide a visual representation of these three demand functional forms. As made evident by the figure, only the portions of the function that satisfy the *Law of Demand* are shown.

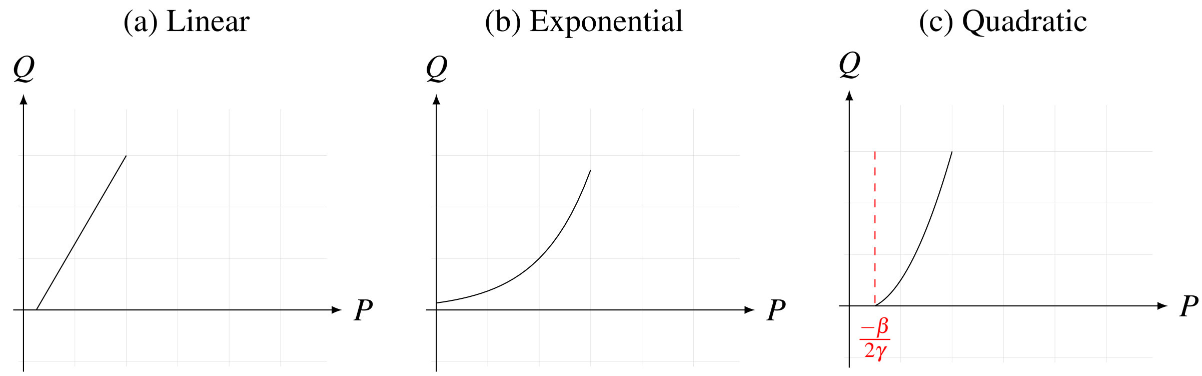


Figure 4.2: Three Functional Forms of the Supply Function

As shown in Figure 4.2, the plotted demand functions do not appear like the supply curve graphs that are typically presented in economics courses. That is, the quantity, Q, is on the y-axis and the price, P, is on the x-axis. The form that you have most likely encountered is the inverse supply. That is,

$$P_i = g(Q_i^S | z_1, z_2, \ldots z_k)$$

To determine the inverse supply, simply solve for price as a function of quantity. Conversely, to derive the supply form, solve for quantity as a function of price.

Firm versus Industry Demand and Supply: It is important to note that there can be important differences between a demand/supply curve that a firm faces and the demand/supply curve for the entire industry.

On the demand side, a firm typically faces a small subset of the consumers that make up an entire industry, and so factors that can impact the firm's demand curve may be different than those that affect the demand curve of the entire industry. For example, consider the demand for paper. While industry demand for paper may be in decline (because of computer technology becoming a cheaper substitute for paper products), a paper sales firm in Scranton, PA may be experiencing greater or lesser impacts of this industry trend as well as local effects. Recognizing the differences between your firm's and the industry's demand curves is critical in correctly responding to changing consumer behavior.

The supply curves are also different for a firm relative to the industry. Even for an industry that produces a relatively homogeneous product (e.g., commodity agriculture), each firm has a different set of inputs. For example, some farms may have higher quality workers or be closer to a chemical supplier. For this farm, the supply curve would be different than the supply curve representing the industry, because the average worker quality or distance to chemical supplier is unlikely to be the same as for the individual farm. Understanding that the supply curve facing your firm may be different than that of the industry can provide important insights about potential comparative advantages and disadvantages of your firm.

There can certainly be exceptions to the rule of thumb that the demand/supply curve faced by the firm is different from that of the industry. For example, one exception can occur when the demand/supply curves facing the firm are nearly identical (most likely by chance) to the industry. This scenario is possible, but unlikely. A more plausible environment where the industry and firm demand/supply curves are similar is in cases where there is a monopoly/monopsony or oligopoly/oligopsony. In these cases, the firm has a significant market share within a sector, and the factors influencing the firm's demand/supply curves would be closely related to affecting the industry's demand/supply curves.

4.3 Equilibrium Conditions

Equilibrium conditions in a market are characterized by the fact that the quantities demanded by consumers are being met by the quantities supplied by producers at a particular price. That is, $Q_i^D = Q_i^S$ and $P_i^D = P_i^S$. Solving for the equilibrium price and quantity in a market requires that you set the demand function equal to the supply function, solve for the equilibrium price, and use the equilibrium to retrieve the equilibrium quantity. Alternatively, you can set the inverse demand function equal to the inverse supply function, solve for the equilibrium quantity, and then use the equilibrium quantity to retrieve the equilibrium price.

Example 4.3.17 Solving for Equilibrium Conditions

Consider the demand function $Q^D = 150 - 10P$ and the corresponding supply function $Q^S = 30 + 5P$. Solve for the equilibrium price and quantity.

$$
\begin{aligned}
Q^D &= Q^S \\
150 - 10P &= 30 + 5P \\
120 &= 15P \\
P_* &= 8
\end{aligned}
$$

$$
\begin{aligned}
Q^D &= Q_* = 150 - 10(8) = 70 \\
Q^S &= Q_* = 30 + 5(8) = 70
\end{aligned}
$$

As expected, regardless of whether the equilibrium price is used to solve for the quantity using the supply or the demand function, the equilibrium quantity is the same.

Practice Problems

1. Consider the demand function $Q^D = 150 - 10P$ and the corresponding supply function $Q^S = 30 + 5P$. Now suppose we introduce an income shifter, such that $Q^D = (150 - 10P)(0.5I)^{0.5}$, where I represents income. Solve for the equilibrium price and quantity when $I = 30$.

2. Using the information above, solve for the equilibrium price and quantity when $I = 50$. Provide economic intuition for the difference between equilibrium price and quantity in the lower and higher income scenarios.

Chapter 5: Demand Theory: Understanding the Consumer

Many managerial economists are asked to evaluated changes in market conditions and assess the best ways to respond. These questions may include:

- How does an improvement in product impact output?
- How does the price of a competitor's good affect the demand for your firm's products?
- If we raise or lower price, how will our sales, revenues, and profits change?
- If expectations about your firm's products' prices change, how will that affect the demand for the products?

A critical concept to grasp is that managers cannot *control* demand. However, if you have a well-developed understanding of your consumers' purchasing behaviors, there is a higher likelihood that you can make decisions to adjust to market changes.

Elasticities are one of the primary tools that managerial economists can use to understand their firm's consumers. There are several advantages of elasticities.

- Elasticities provide a unitless measure for assessing consumers' responsiveness to changes in prices.
- Elasticities offer a method for understanding changes in consumers' purchasing behavior in response to changes in exogenous factors that impact the demand function.
- Elasticity estimates are readily available for many industries, providing managerial economists with ample information for determining market responsiveness.

In general, consider a function $y = f(x)$. The elasticity can be determined as

$$\varepsilon_{y,x} = \frac{dy}{dx}\frac{x}{y} = \frac{dy/y}{dx/x} = \frac{\%\Delta y}{\%\Delta x}$$

Example 5.0.18 A Simple Elasticity Example

Consider the formula for calculating the area of a square, $A = l^2$, where l represents the length of the square side. How much does the area change by if the length of the square sides increases by 1%?

$$
\begin{aligned}
\varepsilon_{A,l} &= \tfrac{dA}{dl}\tfrac{l}{A} \\
&= 2l\tfrac{l}{l^2} \\
&= 2\% \\
\tfrac{\%\Delta A}{\%\Delta l} &= \tfrac{2\%}{1\%}
\end{aligned}
$$

For every 1% change in the length of the sides of the square, the area changes by 2%.

5.1 Price Elasticity of Demand

The price elasticity of demand provides insight about the sensitivity of consumers' purchasing behavior in response to changes in prices. When consumers are sensitive to changes in prices, small increases or decreases in a firm's pricing strategy can result in large changes in sales. Conversely, if consumers are insensitive to price changes, firms can maintain similar sales volumes while changing prices.

The price elasticity of demand answers the question: How much does the quantity demanded change in response to a change in price? Mathematically, this is characterized for good i as

$$\varepsilon_D = \frac{dQ_i^D}{dP_i}\frac{P_i}{Q_i} = \frac{\%\Delta Q_i^D}{\%\Delta P_i}$$

Example 5.1.19 Elasticity for a Linear Demand Function

Consider a linear demand function, $Q = 100 - 2P$. To solve for the elasticity, we need to solve for the slope and the ratio of price to quantity.

Slope: $\frac{dQ}{dP} = -2$

Elasticity: $\varepsilon_D = -2\frac{P}{Q} = \frac{-2P}{100-2P}$

The above example provides two insights about the relationship between the slope and elasticity for a linear demand function.

1. The slope of a linear demand function is ***constant***.
2. The elasticity of a linear demand function is ***not constant***.

Figure 5.1 shows that the absolute value of the elasticity increases when moving down the linear demand curve. Inversely, the absolute value of the elasticity decreases when moving down the inverse linear demand curve.

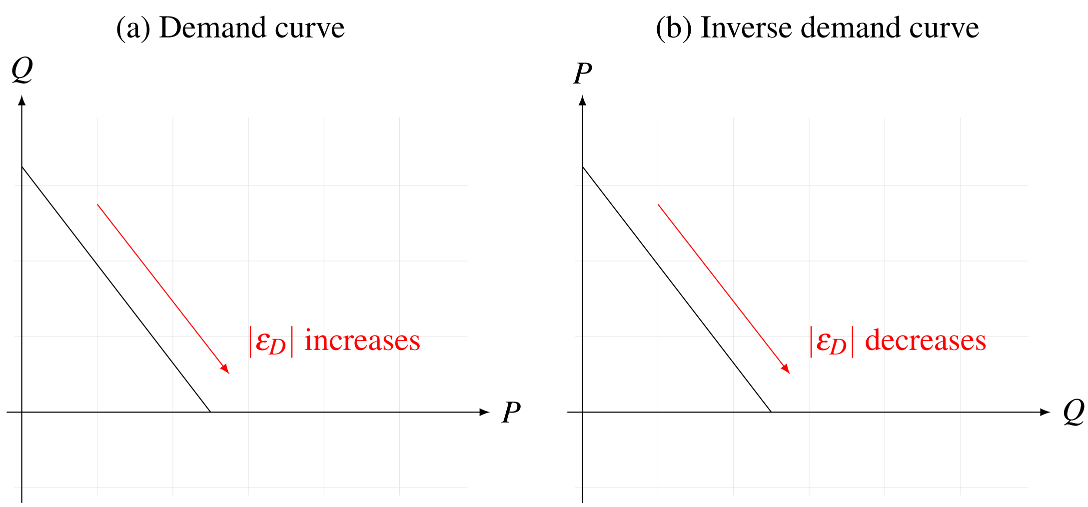


Figure 5.1: Non-constant Price Elasticity of Demand Along a Linear Curve

Example 5.1.20 Elasticity for a Exponential Demand Function

Consider a exponential demand function, $Q = 10P^{-0.5}$. To solve for the elasticity, we need to solve for the slope and the ratio of price to quantity.

Slope: $\frac{dQ}{dP} = -5P^{-1.5}$

Elasticity: $\varepsilon_D = -5P^{-1.5}\left(\frac{P}{Q}\right) = \frac{-5P^{-0.5}}{10P^{-0.5}} = -0.5$

The above example provides two insights about the relationship between the slope and elasticity for a exponential demand function.

1. The slope of a exponential demand function is ***not constant***.
2. The elasticity of a exponential demand function is ***constant***.

In general, for exponential demand functions of the form $Q = \alpha P^{-\beta}$, the elasticity is $\varepsilon_D = -\beta$.

Factors Affecting the Price Elasticity of Demand: There are a number of factors that can affect the magnitude of the price elasticity of demand. That is, factors that can impact the degree to which the measure is elastic or inelastic. The product that your firm is marketing can have a substantial impact in the elasticity of its demand curve.

- Substitutability of other products—greater substitutability will make the demand curve be more elastic.
- Product durability—the demand for more durable products is more elastic.
  - The purchase of durable products is more likely to be postponed.
  - More durable products are also more likely to be repaired.
  - Durable products often have a large used market.
- Proportion of total expenditure that the product has within the overall set of expenses—if a product constitutes only a small portion of the total set of products, the elasticity for that product is likely lower.

- Time period—demand curves are frequently more inelastic in the short run and become more elastic when purchasing decisions can be allocated across longer time periods.

5.1.1 Long-run versus Short-run Elasticity

The time period for which the price elasticity of demand is analyzed is important. Often, there can be significant differences between the price elasticity of demand determined for a short-run time horizon and those determined for a longer-run horizon. As a managerial economist, it is important to recognize that these differences exist.

Estimated Elasticities Over Different Time Horizons

Good/Service	Short-run ε_D	Long-run ε_D
Gasoline	-0.2	-0.7
Natural gas	-0.1	-0.5
Airline travel	-0.1	-2.4
Automobiles	-1.35	-0.2
Tires	-0.9	-1.2

Why do these differences exist? In the short run, consumers and markets are less flexible and, therefore, consumption behaviors are relatively insensitive to changes in prices. In the long run, however, consumers are likely to have greater flexibility in adjusting purchasing behaviors over an extended time period. Additionally, markets are more likely to adjust in the long run, with new competitors entering or exiting a market and the market size and product differentiation changing.

Figure 5.2 provides a visual depiction of the difference between the short run and long run elasticity measures. The figure shows that $|\frac{dQ_{LR}}{dP}| \geq |\frac{dQ_{SR}}{dP}|$. Therefore, if we evaluate the elasticity measures at Q_0 and P_0, then $|\varepsilon_{LR}| \geq |\varepsilon_{SR}|$.

5.2 Recovering Demand Functions from Elasticity Estimates

Elasticities are useful tools. They can be used to evaluate responsiveness measures, as discussed above. However, they are also useful in recovering demand functions. Information about demand

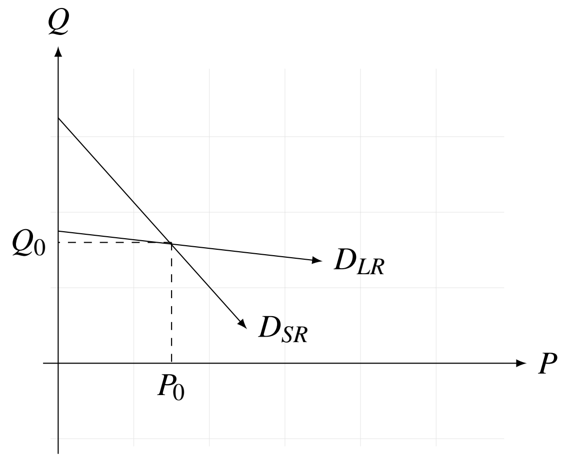


Figure 5.2: Short-run versus Long-run Demand Curves

functions (especially industry demand functions) is frequently difficult to obtain on its own. However, having at least some knowledge about the demand function can be very useful when making managerial decisions.

Empirical estimates of elasticity measures are widely available. For example, for agricultural and food products, the United States Department of Agriculture has a large, searchable web-based database that contains short-run and long-run elasticity estimates across many countries. Using these estimates, you can retrieve an underlying demand function.

In general, suppose that you wish to recover a linear demand function that has the form $Q = \alpha - \beta P$, where α and β are the unknown intercept and slope of that function. That is, you wish to determine the values of α and β. If you know the elasticity value and the price and quantity, then you can first derive β as,

$$\beta = \varepsilon \cdot \frac{Q_0}{P_0}$$

and then α as,

$$\alpha = Q_0 + \beta P$$

In the case of an exponential demand function, $Q = \alpha P^{\beta}$,

$$\begin{aligned} \beta &= \varepsilon \\ \alpha &= \tfrac{Q_0}{P^{\beta}} \end{aligned}$$

Example 5.2.21 Recovering a Linear Demand Function

Suppose that you know that $\varepsilon = -0.8$, $P_0 = 5$, $Q_0 = 100$. Then,

$$\begin{aligned} -\beta &= -0.8\tfrac{100}{5} = -16 \\ \alpha &= 100 + 16(5) = 180 \\ Q &= 180 - 16P \end{aligned}$$

Example 5.2.22 Recovering an Exponential Demand Function

Suppose that you know that $\varepsilon = -0.8$, $P_0 = 5$, $Q_0 = 100$. Then,

$$\begin{aligned} -\beta &= -0.8 \\ \alpha &= 100/(5^{-0.8}) = 362.39 \\ Q &= 363.39P^{-0.8} \end{aligned}$$

5.3 Relationship between Demand Elasticities and Revenue

We have already seen how the elasticity measure can be used to make inferences about consumers' sensitivity to price changes. Firms, however, care more about how their pricing decisions are likely to effect revenues. Having knowledge about elasticities provides an opportunity to do so directly.

Consider that total revenue is the product of total sales (which is a function of price) and price per unit; that is, $TR = Q(P) \cdot P$. We can then determine how total revenue changes in response to price by looking at the appropriate first-order condition:

$$\frac{dTR}{dP} = \frac{dQ}{dP}P + q$$

Recall from the definition of an elasticity that $\varepsilon = \frac{dQ}{dP}\frac{P}{Q}$. Therefore, $Q\varepsilon = \frac{dQ}{dP}P$. Substituting this term in the first-order condition yields

$$\begin{aligned} \tfrac{dTR}{dP} &= Q\varepsilon + Q \\ &= Q(1+\varepsilon) \end{aligned}$$

To determine the elasticity at which total revenue is maximized, we set $\frac{dTR}{dP} = 0$ and solve for ε.

$$\begin{aligned} \frac{dTR}{dP} &= Q(1+\varepsilon) = 0 \\ 1+\varepsilon &= 0 \\ \varepsilon_* &= -1 \end{aligned}$$

This implies that total revenue maximized at the point when the own-price elasticity of demand is unit elastic. That is, when a change in price has a proportionally equal change in quantity demanded.

What does this imply about management decisions and their impacts on total revenue? If you have an estimate of your demand curve or your elasticity value, you can determine whether you operate on the elastic or inelastic portion of that demand curve.

- If you are operating on the inelastic portion, then increasing price would increase revenue, because the resulting decrease in quantity demanded will be proportionally less than the increase in price.
- If you are operating on the elastic portion, then decreasing price would increase revenue, because the resulting increase in quantity will be proportionally greater than the decrease in price.

Figure 5.3 provides a visual characterization of the above properties. The figure shows that the pricing decisions depend on the point of the demand curve at which your firm is currently operating. The objective is to alter the firm's pricing strategy such that the firm moves toward a price/quantity combination at the unit elastic portion of the curve.

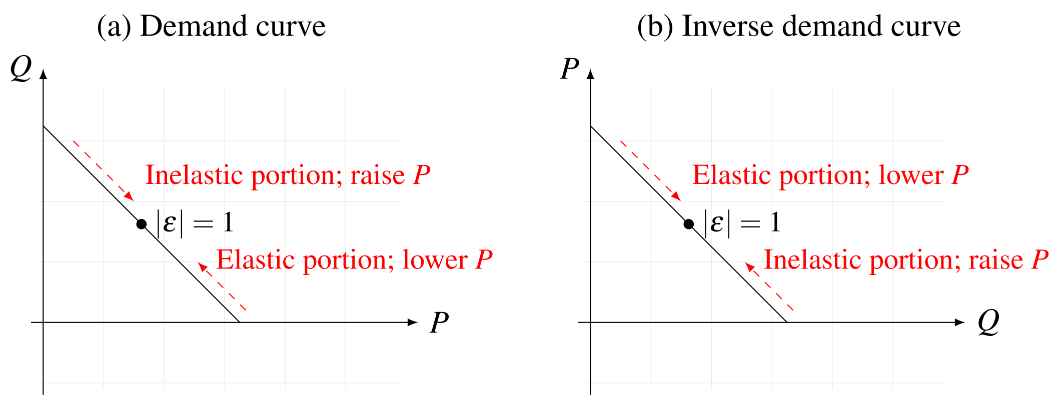


Figure 5.3: Relationship between Price Elasticity of Demand and Revenue

Practice Problem

You are operating a small snow plowing business. To plow a typical residential block, you charge \$50. At this price, you plow 100 blocks. You estimate that the demand curve for snow plowing is $Q = 125 - 0.5P$.

1. Determine whether you are operating on the elastic or inelastic portion of the demand curve.
2. Should you increase or decrease your price to increase total revenue?
3. Show whether your prediction in (2) is correct by calculating your total revenues at $P_0 = \$50$ and $P_1 = P_0 \pm 10$.
4. What is the optimal price at which total revenues are maximized?

5.4 Other Elasticities of Demand

So far, we have only considered *ceteris paribus* conditions when examining the own-price elasticity of demand. That is, we have examined the changes in quantities demanded in response to variation in price, but we have done so under the assumption that no other factor changes.

The same approach can be used to assess quantity demanded responses when there are changes in other, exogenous factors, such as the price of other goods or incomes. As with the own-price elasticity of demand, these changes are evaluated under the assumption that the price of the good remains the same.

5.4.1 Income Elasticity of Demand

Consider the general demand function, $Q_i^D = f(P_i|I, x_1, \dots x_k)$, where I represents consumers' income levels. The income elasticity of demand is

$$\varepsilon_{Q,I} = \left.\frac{dQ_i}{dI}\right|_{P_i=P_0}\left(\frac{I}{Q_{i,0}}\right)$$

where the term $|_{P=P_0}$ indicates that we evaluate the income elasticity of demand while holding the own price of the good constant at $P_i = P_0$.

Example 5.4.23

Consider the demand function $Q = 0.1(I/P)$. Calculate the income elasticity of demand.

$$\begin{aligned}\varepsilon_{Q,I} &= (0.1/P)\tfrac{I}{Q} \\ &= \tfrac{0.1}{P}\tfrac{I}{0.1(I/P)} = 1\end{aligned}$$

Therefore, for a 1% change in consumers' income, there is a 1% change in the quantity demanded at the current price.

Example 5.4.24

Consider the demand function $Q = 100 + 0.4I - 0.5P$. Calculate the income elasticity of demand at $I = 200$ and $P = 6$.

First, solving $Q = 100 + 0.4(200) - 0.5(6) = 177$. Next, solve for the income elasticity of demand.

$$\begin{aligned}\varepsilon_{Q,I} &= 0.4\left(\tfrac{200}{177}\right) \\ &= 0.45\%\end{aligned}$$

Therefore, for a 1% change in consumers' income, there is a 045% change in the quantity demanded at the current price of $P = 6$.

5.4.2 Cross-price Elasticity of Demand

Consider the general demand function, $Q_i^D = f(P_i|I, P_j, x_1, \ldots x_k)$. This demand function is dependent on the own price, P_i, the income, I, price of another good, P_j, and other factors. At this point, noting subscripts becomes critical for maintaining order in your calculations. The cross-price elasticity of demand is

$$\varepsilon_{Q_i,P_j} = \frac{dQ_i}{dP_j}\bigg|_{P_i=P_0}] \left(\frac{P_j}{Q_{i,0}}\right)$$

Example 5.4.25

Consider the demand function $Q_i = 70 - 0.2P_i + 0.1P_j + I$. Calculate the cross-price elasticity of demand.

$$\begin{aligned} \varepsilon_{Q_i,P_j} &= 0.1\left(\frac{P_j}{Q_i}\right) \\ &= 0.1\left(\frac{P_j}{70-0.2P_i+0.1P_j}\right) \end{aligned}$$

Therefore, the elasticity depends on the current values of $P_i,\ P_j$, and I.

Example 5.4.26

Consider the demand function $Q_i = 10P_i^{-0.4}P_j^{0.2}I$. Calculate the cross-price elasticity of demand.

$$\begin{aligned} \varepsilon_{Q_i,P_j} &= 2P_i^{-0.4}P_j^{-0.8}I\left(\frac{P_j}{10P_i^{-0.4}P_j^{0.2}I}\right) \\ &= 0.2 \end{aligned}$$

Therefore, the elasticity is constant and corresponds to the power coefficient associated with P_j in the demand function.

5.5 Comparative Statics of Equilibrium Conditions

Elasticities can be used to evaluate not only individual demand curves but also equilibrium market conditions. That is, they can help assess how much equilibrium prices increase or decrease and how much equilibrium quantities increase or decrease when moving from one equilibrium condition to another. Figure 5.4 shows that moving from equilibrium E_0 to E_1 results in a change in price and a change in quantity. We may be interested in answering the questions: What was the percent change in prices when moving from P_0 to P_1? What was the percent change in quantity when moving from Q_0 to Q_1?

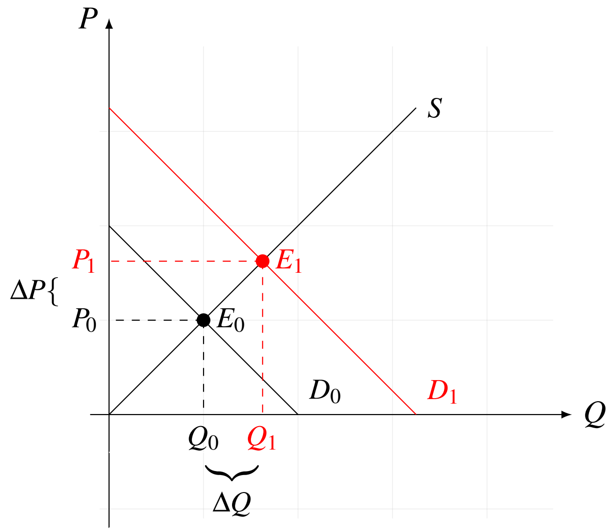


Figure 5.4: Changes in Equilibrium Conditions

Example 5.5.27 Comparative Statics

Consider the supply and demand functions $Q^S = 10(P/w)$ and $Q^D = 100 - 2P$, where w represents wage costs. Determine the change in the equilibrium price and quantity values from a 1% change in wage costs.

First, solve for the equilibrium conditions.

$$\begin{aligned}
10(P/w) &= 100 - 2P \\
10(P/w) + 2P &= 100 \\
P(10/w + 2) &= 100 \\
P_* &= \frac{50w}{5+w} \\
Q_* &= 10\left(\frac{50w}{5+w}\right)\left(\frac{1}{w}\right) = \frac{500}{5+w}
\end{aligned}$$

Now that the equilibrium price and quantity conditions are determined, solve for the elasticities of each with respect to changes in the wage costs.

$$\begin{aligned}
\varepsilon_{P_*,w} &= \frac{dP_*}{dw}\frac{w}{P_*} \\
&= \frac{50(5+w)-50w}{(5+w)^2}\frac{w}{P} \\
&= \left(\frac{250}{(5+w)^2}\right)\left(\frac{5+w}{50w}\right)w \\
&= \frac{5}{5+w}
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{Q_*,w} &= \frac{dQ_*}{dw}\frac{w}{Q_*} \\
&= \left(\frac{-500}{(5+w)^2}\right)\left(\frac{5+w}{500}\right)w \\
&= \frac{-w}{5+w}
\end{aligned}$$

Both elasticities depend on the wage costs, w. Suppose that we wish to evaluate the elasticities at $w = \$7.00$. Then,

$$\begin{aligned}
\varepsilon_{P_*,w} &= \frac{5}{13} = 0.38\% \\
\varepsilon_{Q_*,w} &= \frac{-7}{13} = -0.54\%
\end{aligned}$$

As expected, when input costs increase, the supply curve shifts inward/left resulting in higher prices and lower quantities supplied. The calculated elasticities reflect these expected changes.

Practice Problem

Consider the supply and demand functions $Q^S = 200P^{0.5}$ and $Q^D = \frac{50I}{P^2}$, where I represents income. Determine the following:

1. Equations for the equilibrium price and quantity.
2. The elasticity of equilibrium price with respect to a change in income.
3. The elasticity of equilibrium quantity with respect to a change in income.

5.6 Modeling Consumers under Uncertainty

Earlier in the course notes, we discussed the basic concept of uncertainty and modeling how individuals and firms can evaluate expected outcomes when facing uncertainty of outcomes. We also considered how different forms of utility functions could be used to characterize more or less risk averse individuals and firms.

Instead of thinking of a particular utility functional form, we can generalize the model in such a way that we can simply alter the degree of risk aversion, but maintain the same utility function. This is especially useful when you don't know and/or are unable to estimate the utility function of your consumers.

5.6.1 Constant Relative Risk Aversion Utility Function

One of the most used utility function form is the constant relative risk aversion (CRRA) utility. For example, for some level of income, I, the CRRA function is

$$U(I)\frac{I^{(1-\omega)}}{(1-\omega)}$$

The level of risk aversion can be easily altered by changing the level of ω. The term is bounded $\omega \in (-1,1)$. A risk neutral individual would have a risk aversion factor $\omega = 0$. More risk averse individuals would be modeled with $\omega > 0$, with higher values characterizing more risk averse individuals. Conversely, risk loving individuals would be modeled with $\omega < 0$.

Example 5.6.28 Modeling Difference Risk Aversion Levels Using the CRRA Function

Assume that an individual's income is $I = 10$. Show differences in utilities for three different risk aversion assumptions: $\omega = 0$, $\omega = 0.5$, and $\omega = -0.5$.

When $\omega = 0$ (i.e., risk neutrality), utility is

$$\begin{aligned} U(I=10) &= \frac{10^{(1-0)}}{(1-0)} \\ &= 10 \end{aligned}$$

When $\omega = 0.5$ (i.e., risk averse), utility is

$$\begin{aligned} U(I=10) &= \frac{10^{(1-0.5)}}{(1-0.5)} \\ &= 6.3 \end{aligned}$$

When $\omega = -0.5$ (i.e., risk loving), utility is

$$\begin{aligned} U(I=10) &= \frac{10^{(1-(-0.5))}}{(1-(-0.5))} \\ &= 21.1 \end{aligned}$$

Comparing the three utilities of income, it is clear that the positive income provides the most utility for the risk loving individual, and least utility for the risk averse.

5.6.2 Arrow-Pratt Risk Aversion Measure

When considering how to compare the risk aversion of different consumers or consumer groups, it is helpful to compare relative risk aversion of the two consumer groups. The Arrow-Pratt measure of risk aversion is a tool used to make this comparison.

The Arrow-Pratt risk aversion measure for a utility of income, $U(I)$ is

$$r(I) = -\frac{U''(I)}{U'(I)}$$

That is, the measure is the negative of the ratio between the second and first derivatives of the utility function. The larger the number, the more risk averse an individual or consumer group is relative to a group with a lower Arrow-Pratt risk aversion measure.

Example 5.6.29 Comparing Risk Aversion using the Arrow-Pratt Measure

Suppose that your firm needs to assess the risk aversion of two consumer groups. Group A is characterized by the utility of income function, $U_A(I) = \ln(I+2)$. Group B has a utility of income function, $U_B(I) = (I+2)$. If you wanted to focus on marketing to the less risk averse group, which group would you select?

To determine the relative risk aversion, we can calculate the Arrow-Pratt risk aversion measure using the utility function of each group. This requires that we first determine the first and second derivatives of each function and then calculate the ratio.

For Group A, the calculation is

$$\begin{aligned} U_A'(I) &= \frac{dU_A}{dI} \\ &= \frac{1}{(I+2)} \end{aligned}$$

$$\begin{aligned} U_A''(I) &= \frac{dU_A'}{dI} \\ &= -\frac{1}{(I+2)^2} \end{aligned}$$

$$\begin{aligned} r_A(I) &= -\frac{U_A''(I)}{U_A'(I)} \\ &= -\left(-\frac{1}{(I+2)^2} / \frac{1}{(I+2)}\right) \\ &= \frac{1}{(I+2)} \end{aligned}$$

For Group B, the calculation is

$$
\begin{aligned}
U_A'(I) &= \frac{dU_A}{dI} \\
&= 1 \\
\\
U_A''(I) &= \frac{dU_A'}{dI} \\
&= 0 \\
\\
r_A(I) &= -\frac{U_A''(I)}{U_A'(I)} \\
&= -(0/1) \\
&= 0
\end{aligned}
$$

Clearly, because $r_A(I) > 0$ for positive values of income, consumer Group A is more risk averse than consumer Group B. Therefore, the strategy should focus on Group B.

End of Chapter Practice Problems

1. Consider that you operate a coffee shop. You sell coffee at a price of \$2.00 per cup and sell 250 cups each day. A tea house across the street sells tea at \$2.50 per cup and sells 200 cups each day. A year ago, you lowered the price \$2.50 per cup to compete with the tea house, and found that you sold an extra 25 cups per day. Even more recently, the tea house raised its price from \$2.25 per cup and lost 25 cups worth of sales. Assume that the customers who purchased those 25 cups of tea before the price increase switched over to purchasing coffee. That is, you experienced an increase of 25 cups of coffee sold as a result of the tea house's price increase.

   (a) Assume that the demand for coffee has an exponential functional form. Solve for the underlying demand curve.

   (b) Assume that the demand for coffee has a linear functional form. Solve for the underlying demand cure.

   (c) In the linear demand curve scenario, determine the revenue-maximizing price.

2. Consider the demand function $Q^D = 250 - 10P + 0.025P^2$. Does the quantity-maximizing price also maximize total revenues?

3. Suppose that you manage a garage door company that sells and installs garage doors for the city's residential housing community. As a manager, you know that individuals' incomes are an important market factor in determining business for your company, because as income increases so will the demand for new housing and for repairs in existing housing. Currently, the company sells 10,000 doors per year at a price of \$1,500 per door. If median income goes up from \$32,000 to \$34,000, you expect to sell 12,000 doors per year. Also, if income does not change but you increase the price of garage doors by \$100, you expect to sell only 8,500 doors per year. Assume that the underlying demand function for garage doors is $Q^D = \alpha - \beta P + \theta I^{0.5}$, where α, β, and θ are the unknown parameters of the demand function. What is the revenue-maximizing price at each income level, $I = \$32000$ and $I = \$34000$?

Chapter 6: Production Theory: Making Firm-level Decisions

Understanding the consumer market is important, because it offers insights into questions such as: Are there individuals who will purchase my good or service? How much are they willing to pay for this good or service and how much will they consume? How do these consumers respond to changes in prices? Knowing that a market has the potential to exist is critical.

As a managerial economist, the next step is to develop an optimal strategy for operating in this market. That is, we want to answer questions such as: What good or service should we produce? How much of it should we produce? What are the necessary inputs and how much of those inputs should we use? How do we respond to changes in how much we receive for a good or service and how much we have to pay to produce it?

In general, ***production*** is the process of combining a set of inputs to create an output. Your goal will be to manage this production process such that the firm's profits are maximized, subject to maximizing consumers' utility through the sale of your good or service. This constrained optimization problem is the foundation of production theory.

Assumptions about production: we will make several assumptions about the production process, which will simplify our discussion.

1. Production is ***wasteless***, in that we obtain maximum possible output from an input.
2. Technology is ***fixed*** and ***known***.
3. Production in period t does not affect production in period $t+1$.
4. All input and output units are ***homogeneous***.
5. There is certainty in the input-to-output relationship.

6.1 Producers and Consumers

We will begin by considering two of the questions that a managerial economist will be tasked with answering: What to produce and how much to produce? In evaluating the answers, we return to the core issue addressed by economics: assessing trade-offs. That is, a firm always has some choices in producing or offering from a set of goods or services. Moreover, because the firm is constrained by its available resources, it can only produce up to a certain quantity and then decide how much of each good to produce.

6.1.1 Production Possibility Frontier

To model the aggregate production opportunities and the output trade-offs, we use the ***production possibility frontier*** (PPF). This is a visual, relatively straightforward model for characterizing these trade-offs. That is, it illustrates the combination of outputs that can be produced given a particular level of inputs, if no inputs are wasted.

Figure 6.1 shows an example of the PPF. Typically, the PPF curve will be concave to the origin and at each point on the curve will represent the maximum possible production of alternative goods. The figure characterizes a simple two-good scenario, in which the curve represents the maximum production for two goods, Q_1^* and Q_2^*.

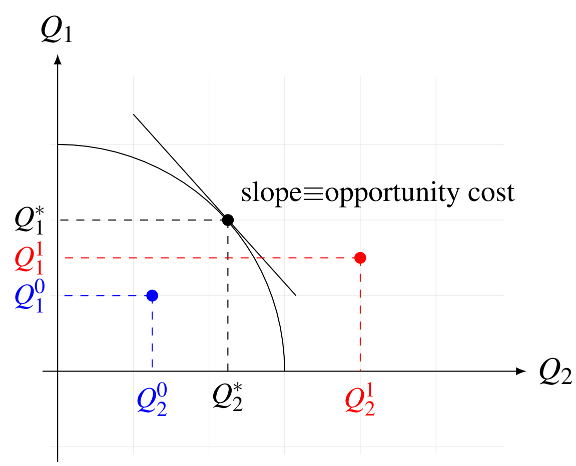


Figure 6.1: Production Possibility Frontier

Points of production that are inside of the curve (Q_1^0 and Q_2^0) represent inefficient output. For example, inputs may be used ineffectively such that more output is possible per unit of input. Alternatively, points of production that are outside of the PPF curve (Q_1^1 and Q_2^1) are unattainable given the existing level of production technology.

Mathematically, the PPF can be described as $Q_1 = f(Q_2)$. This function explicitly characterizes the trade-offs among producing the two goods, Q_1 and Q_2, because the production of one depends on the level of output of the other. Explicitly, this dependence can be represented by taking the derivative of th function with respect to Q_2; that is, $\frac{dQ_1}{dQ_2}$. This derivative represents the slope at the point at which the slope is evaluated. This trade-off is known as the ***marginal rate of product transformation*** (MRPT).

In Figure 6.1, this trade-off is represented by the line tangent to the PPF at the point (Q_1^*, Q_2^*). The economic interpretation of the slope of the line is the ***opportunity cost*** of producing one additional unit of Q_2. That is, it shows how much fewer units of Q_1 you will produce if you began to produce one additional unit of Q_2. For example, if the $\frac{dQ_1}{dQ_2} = -\frac{2}{1}$, then at the current level of output, producing one more unit of Q_2 would result in a two unit reduction of good 1.

Example 6.1.30 Evaluating Opportunity Costs Using the PPF

Consider two small business owners, an economist who offers economic analysis services and a landscaper who provides hedging services. The economist needs some landscaping done while the landscaper needs to evaluate his business. Each person is evaluating whether they should do each of services themselves or hire the other person. They can assess their opportunity costs by considering the PPF and the slope at the point of their current operations.

Figure 6.2 shows that for the economist (left figure), the slope is $-\frac{1}{2}$. That is, she would have to give up two economics projects to do her own landscaping. For the landscaper, the slope is $-\frac{2}{1}$, which would require him to give up two landscaping projects to do his economic analysis. The opportunity costs of switching to another activity are higher for both individuals.

6.1.2 Characterizing Consumption Behavior

Before continuing with production theory, we must briefly return to consumers. This is due to the constrained optimization foundation of a firm's production decisions, which we discussed above. That is, the firm must maximize profits, but do so by maximizing the utility of its consumer. Therefore, we will need to understand when a consumer's utility is maximized, and only then consider profit maximization.

Consumption choices are described using ***indifference curves***. These curves represent the trade-offs in utility when choosing to consume different quantities of two or more goods. Each indifference curve represents the same utility level, regardless of how much of each good is

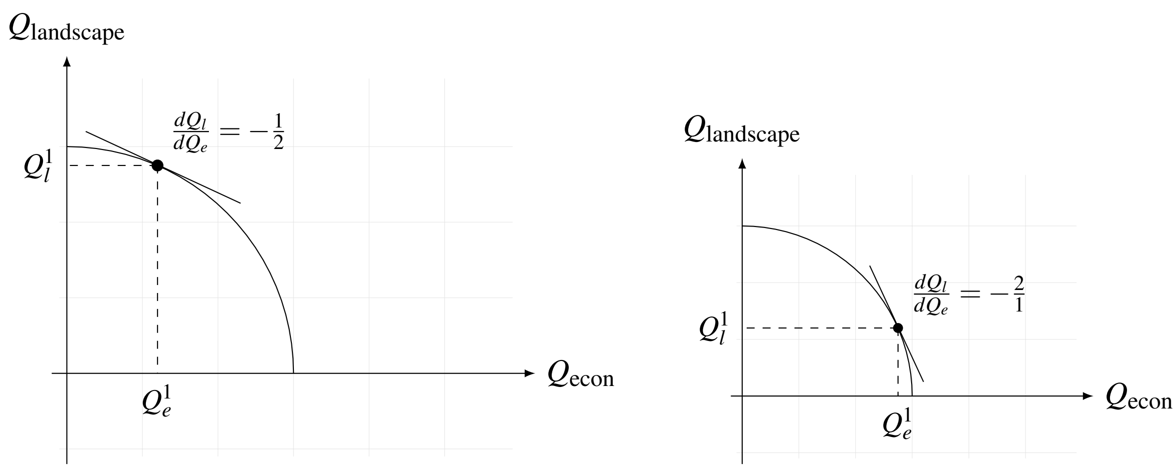
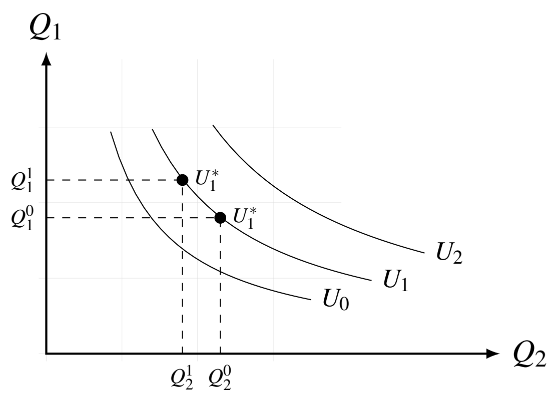


Figure 6.2: Comparison of Opportunity Costs

consumed. Higher utility curves (further from the origin) represent greater utility levels and indifference curves never intersect.

Figure 6.3 shows an example of indifference curves for an individual who is consuming different quantities of two goods, Q_1 and Q_2. The figure presents three instances (out of an infinite number) of indifference curves, with U_0 representing the lowest utility level and U_2 representing the highest.

Figure 6.3: Indifference Curves



All of the curves are convex to the origin and each curve represents different quantity consumption combinations of the two goods that would lead to the same utility level. For example, the individual has the same utility level, U_1^*, regardless of whether she is consuming the combination (Q_1^0, Q_2^0) or the combination (Q_1^1, Q_2^1).

For a two good case, the indifference curve is characterized by the equation, $U = U(Q_1, Q_2)$. At any point on the indifference curve (i.e., at any combination of consumption), the line tangent to the point is $\frac{dQ_1}{dQ_2}$. Similar to the PPF, this slope describes the amount of good 1 that must be given up in order to consume another unit of good 2, such that the utility remains the same. This is known as the ***marginal rate of substitution*** (MRS).

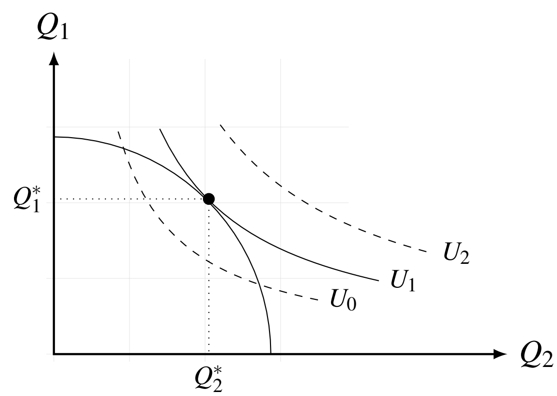
6.1.3 Putting It All Together: Non-market Case

In the non-market case, there are no prices. The producer and consumer have the same objective: maximize consumer's utility in the most efficient way. That is, it is necessary to determine the optimal quantity combination of two goods to maximize utility, subject to the production capabilities of the producer. Mathematically,

$$\max U(Q_1, Q_2)$$
subject to
$$Q_1 = f(Q_2)$$

Figure 6.4 provides a visual characterization of the non-market scenario. The figure shows that the optimal point of production is where the PPF is tangent to the highest possible indifference curve. That is, if the consumer was on the indifference U_0, they are inside of the PPF and are not maximizing the possible consumption. If the consumer wished to be on the indifference curve U_2, they would be outside of the possible range of production. The optimal scenario is being on the U_1 indifference curve.

Figure 6.4: Production in a Non-market Environment



At the optimal production and consumption point, the slope of the PPF is equal to the slope of the indifference curve.

$$\text{MRPT} = \text{MRS}$$
$$\frac{dQ_1}{dQ_2}_{\text{PPF}} = \frac{dQ_1}{dQ_2}_{\text{Indiff}}$$

6.1.4 Putting It All Together: Market Case

While the non-market environment is relatively straightforward, the more realistic scenario is one in which prices exist. In this environment, we make the following familiar assumptions:

1. Producers are profit maximizing.
2. Producers can choose to output two products of quantities Q_1 and Q_2.
3. Producers are price takers, with prices P_1 and P_2 for the two goods.

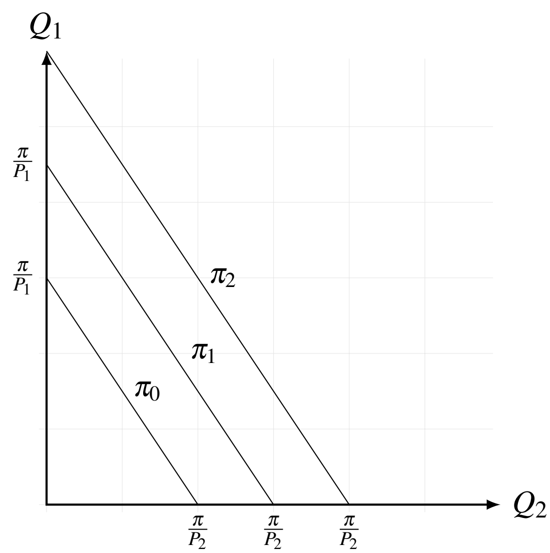
Generally, the profit function is defined as

$$\pi = f(P_1Q_1, P_2Q_2)$$

A simple linear representation of the profit function is, $\pi = P_1Q_1 + P_2Q_2$.

The linear profit function can be re-written as $Q_1 = \frac{\pi}{P_1} - \frac{P_2}{P_1}Q_2$ and visually represent this function under different price scenarios. Figure 6.5 presents this visualization. For each of the lines, the slope is $-\frac{P_2}{P_1}$, which indicates that to stay on the same profit curve, if the price of one good increases, the price of the other good has to decrease.

Figure 6.5: Production in a Non-market Environment

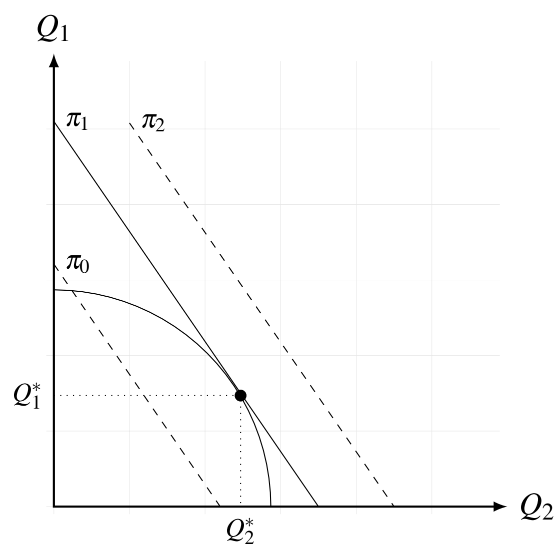


Intuitively, given a set of prices, producers wish to be on the highest profit line. However, they must do so within the constraint of the production possibility. Mathematically,

$$\max \pi = P_1 Q_1 + P_2 Q_2$$
subject to
$$Q_1 = f(Q_2)$$

Figure 6.6 presents the optimal conditions for the producer in the market scenario. The producer could operate on the profit like π_0, but it would be inefficient because they can produce and sell more units. The producer wishes to be on π_2, but cannot do so because it is outside of the possible production range. Therefore, producers operate at the point where they can produce the maximum quantity and operate on the highest profit line at that production level.

Figure 6.6: Production in a Non-market Environment



Mathematically, we ask the question: Given some set of prices and some production quantity Q_1, what is the optimal quantity Q_2 to maximize profit?

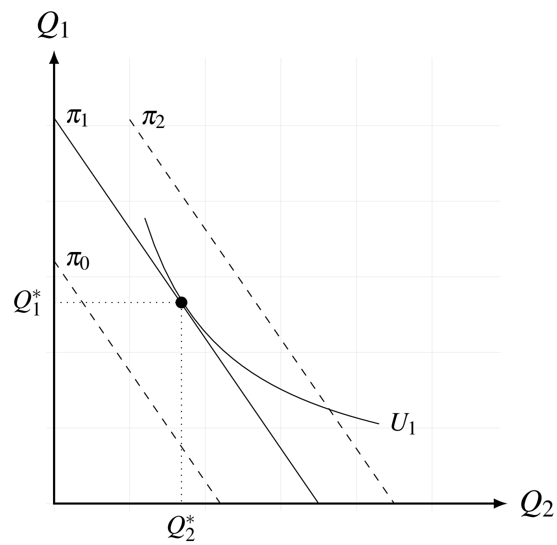
$$\frac{d\pi}{dQ_2} = P_1 \frac{dQ_1}{dQ_2} + P_2 = 0$$
$$\frac{dQ_1}{dQ_2} = -\frac{P_2}{P_1}$$

The equality shows that at the profit maximizing point, the marginal rate of production transformation equals the negative ratio of prices. That is, the point where the PPF is tangent to the highest possible profit line, with the slope $-\frac{P_2}{P_1}$.

Now consider a consumer in the market scenario. In the simplest case, consider a single consumer who is also a producer in a market. The profits that are earned by the individual when wearing the "producer hat" become the available budget for the individual when wearing the "consumer hat." Therefore, the profit curve for the consumer is treated as a budget constraint.

Consumers wish to maximize their utility, subject to a budget constraint. Figure 6.7 provides a visual representation of this problem. The logic is similar to the producer's case. Mathematically,

Figure 6.7: Production in a Non-market Environment



the problem is set up as follows:

$$\max U = U(Q_1, Q_2)$$
subject to
$$\pi = P_1 Q_1 + P_2 Q_2$$

We can re-write the profit function as $Q_1 = \frac{\pi}{P_1} - \frac{P_2}{P_1} Q_2$ and then substitute this equation into the utility function. This allows us to re-specify the maximization problem without an explicit constraint and with respect to only one good. That is,

$$\max U = U\left(\frac{\pi}{P_1} - \frac{P_2}{P_1} Q_2, Q_2\right)$$

From here, we can solve for the first-order condition $\frac{dU}{dQ_2} = 0$. This will yield the result

$$\frac{dQ_1}{dQ_2} = -\frac{P_2}{P_1}$$

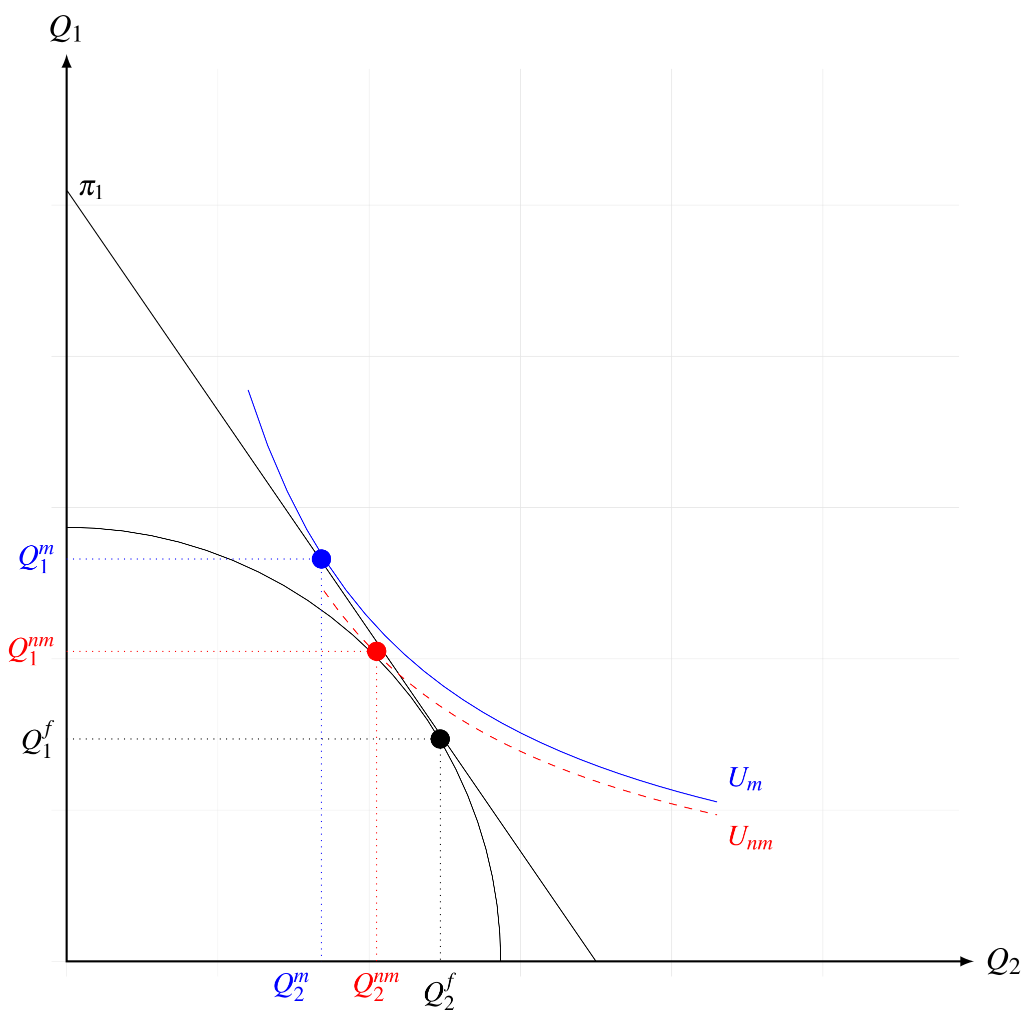
This result states that the optimal consumption behavior is dependent on the ratio of prices, which is the slope of the profit curve. That is, at the optimal consumption point, the indifference curve is tangent to the profit curve.

6.1.5 Separation Principle

In the previous section, we considered the optimal production and consumption strategies in both the non-market and market cases. Here, we will consider how these strategies (and resulting behaviors) differ. The outcome yields a critical concept: the ***separation principle***.

Figure 6.8 provides a visual comparison of the market and non-market scenarios. In the non-market scenarios, both the producer outputs and the consumer consumes Q_1^{nm} and Q_2^{nm}. In the market scenario, the producer outputs quantities Q_1^f and Q_2^f and the consumer consumes quantities Q_1^m and Q_2^m.

Figure 6.8: Production in a Non-market Environment



There are two critical inferences from analyzing the figure.

1. Consumers' utility is higher in the market scenario, $U^m \geq U^{nm}$.
2. Producers' only goal is to maximize profits, which *does not* depend on knowing the shape of consumers' utility function.

Separation Principle: Producers can *separate* their production decisions from knowledge about their consumer's exact utility functions. In maximizing profits rather than the utility of any individual consumer, firms operate on their PPF (i.e., optimally) and consumers achieve a higher utility.

6.2 The Production Function

The separation principle indicates that firms can focus on maximizing profits, rather than needing to considering independent utility functions of their consumers. Therefore, we will focus on firms' decisions. The foundation of this focus is the ***production function***.

To develop the fundamental concepts, we will consider a single input production function,

$$Q = f(x)$$

where Q represents the quantity produced, x is the quantity of inputs, and $f(\cdot)$ represents the function of converting inputs into outputs. Figure 6.9 presents a visual characterization of a classic production function.

In the classic production function, the greatest response (steepest positive slope) occurs at lower levels of input. In the classic production, which is characteristic of all other production functions, is that at least some part of the function has to have a positive slope. Otherwise, there would be no reason to increase the use of inputs.

Figure 6.10 shows two examples of alternative representations of production functions. In the left sub-figure, the production function is characterized by an input conversion process that always yields positive returns (i.e., the slope is never less than zero), but there are decreasing marginal returns to additional inputs. In the right sub-figure, the production function is piecewise. There is an initial positive portion with constant returns to additional inputs, but at some point, adding more inputs yields no additional outputs (i.e., inputs are wasteful, albeit not detrimental).

Figure 6.9: Classic Production Function

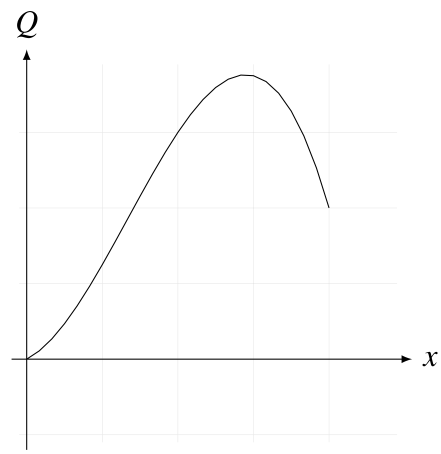
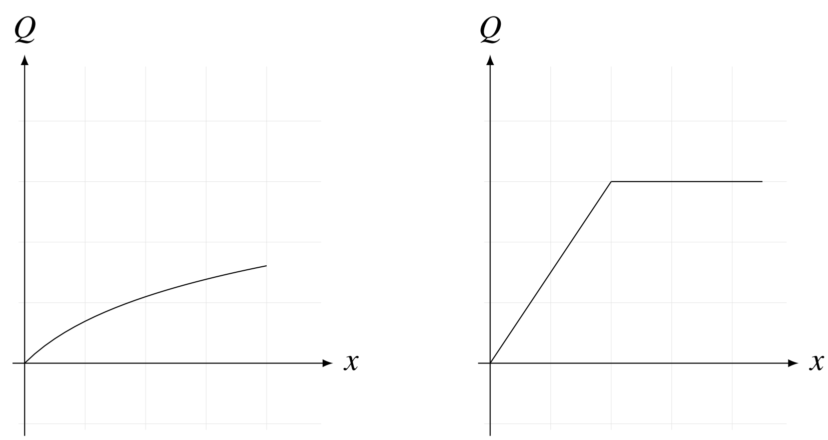


Figure 6.10: Alternative Production Function Representations



6.2.1 *Economic Properties of Production Functions*

It is useful to describe the mathematical and economic properties of a production function. We will consider three basic aspects: the total product, average product, and marginal product.

Total product
The total product is the total output produced at a given level of input. The total product is characterized by the equation $Q = f(x)$.

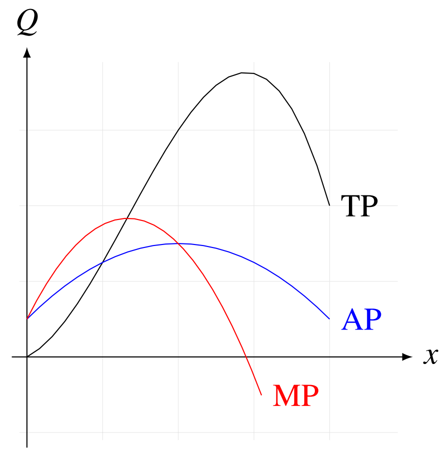
Average product
The average product is the average output produced per unit of input. The average product is characterized by the equation $\text{AP} = \frac{Q}{x} = \frac{f(x)}{x}$.

Marginal product

The marginal product is the additional units of output produced per unit of input. The marginal product is characterized by the equation $\text{MP} = \frac{dQ}{dx} = \frac{df(x)}{dx}$. Mathematically, the marginal product is the slope of the production function at any level of inputs. The marginal product typically declines over some range and it is equal to zero at the point where production is maximized.

Figure 6.11 presents the visual representation of the total, average, and marginal products.

Figure 6.11: Total, Average, and Marginal Products



Example 6.2.31 Solving for Average and Marginal Products

Consider the total product function $Q = x + 0.5x^2 - 0.05x^3$. The average and marginal products are

$$\begin{aligned} \text{AP} &= \frac{Q}{x} \\ &= \frac{x+0.5x^2-0.05x^3}{x} \\ &= 1 + 0.5x - 0.05x^2 \end{aligned}$$

$$\begin{aligned} \text{MP} &= \frac{dQ}{dx} \\ &= 1 + x - 0.15x^2 \end{aligned}$$

Relationship Between Average and Marginal Products

Consider that the average product can be written as, $\text{AP} = Q(x)x^{-1}$. Taking the derivative of the average product with respect to x.

$$\begin{aligned} \frac{dAP}{dx} &= \frac{dQ}{dx}x^{-1} - Q(x)x^{-2} \\ &= \frac{1}{x}\left(\frac{dQ}{dx} - \frac{Q}{x}\right) \\ &= \frac{1}{x}(\text{MP} - \text{AP}) \end{aligned}$$

This provides three inferences:

1. If MP>AP, then $\frac{dAP}{dx} > 0$.

2. If MP<AP, then $\frac{dAP}{dx} < 0$.

3. If MP=AP, then $\frac{dAP}{dx} = 0$. That is, it is the level of x where average product is maximized.

6.2.2 Stages of Production

As a managerial economist, you will be tasked with making optimal production decisions. That is, if you know the production function, you could decide on any output quantity across the function. But what is the optimal output quantity?

<u>Mechanism</u>

1. Determine output, Q, where $x = 0$.
2. Determine the level of x and Q where the average product is maximized (i.e., MP=AP).
3. Determine the level of x and Q where output is maximized (i.e., MP=0).

After determining these quantities, specify the stages of production.

- **Stage I**: Ranges from $(x = 0,\ Q(x = 0))$ to $(x, Q(x))_{\text{AP is max}}$.
- **Stage II**: Ranges from $(x, Q(x))_{\text{AP is max}}$ to $(x, Q(x))_{\text{Q is max}}$.
- **Stage III**: Input quantities beyond $(x, Q(x))_{\text{Q is max}}$.

Optimal Production Decision

- In Stage I, output returns to an additional unit of input are positive and increasing. Therefore, the firm can obtain additional output per unit relative to what was obtained from the previous unit.
- In Stage II, output returns to an additional unit of input are positive but decreasing. Therefore, the firm can obtain additional output per unit of input, but the additional output is less than what was obtained from the previous unit of input.
- In Stage III, returns to an additional unit of input are negative.

Therefore, the optimal region of production is Stage II.

Example 6.2.32

Consider the production function $Q = x + 0.5x^2 - 0.05x^3$. Determine the three production stages.

1. Determine output where $x = 0$.

$$Q(x = 0) = 0$$

2. Determine input and output quantities such that average product is maximized.

$$\begin{aligned}
\text{AP}: \quad & \frac{Q}{x} \quad 1 + 0.5x - 0.05x^2 \\
\frac{d\text{AP}}{dx} &= 0.5 - 0.1x = 0 \\
&= x^* = 5 \\
Q(x = 5) &= 11.25
\end{aligned}$$

3. Determine input and output quantities such that quantity is maximized.

$$\frac{dQ}{dx} = \text{MP} = 1 + x - 0.15x^2$$

$$x^* = \frac{-1 \pm \sqrt{1^2 - 4(-0.15 \times 1}}{2(-0.15)}$$

$$x^* = (-0.88, 7.55)$$

The only reasonable value of inputs is $x^* = 7.55$. At this input quantity, $Q = 12.78$.

Production Stages

Stage I: From $(x = 0, Q = 0)$ to $(x = 5, Q = 11.25)$
Stage II: From $(x = 5, Q = 11.25)$ to $(x = 7.55, Q = 12.78)$
Stage III: After $(x = 7.55, Q = 12.78)$

Practice Problem

Consider the production function $Q = 5x - x^2$. Graph the total, average, and marginal products. Then, determine the stages of production and comment on the optimal production decision.

Production Functions with Intercepts

The classic production function and variations of the classic function models the conversion of inputs into outputs such that if $x = 0$, then $Q = 0$. That is, this class of production functions does not have an intercept.

However, there are situations when this is not the case. A classic example is crop production. Suppose that you plant a seed and then expend no other inputs (e.g., no fertilizer, no irrigation, etc.) It is likely that the seed will still produce a plant from which you have some output. In this case, the related production function will need to have a positive intercept to correctly characterize the fact that when $x = 0$, $Q > 0$.

When dealing with these types of production functions, it is necessary to consider whether the production stages can continue to be calculated as with classic production functions. That is, it is necessary to ask whether Stage II begins at the point where MP=AP.

Let's consider four production functions.

$$
\begin{aligned}
Q_{\beta_0=0} &= x+0.1x^2-0.01x^3 & \rightarrow\ AP_{\beta_0=0} &= 1+0.1x-0.01x^2 \\
Q_{\beta_0=0.1} &= 0.1+x+0.1x^2-0.01x^3 & \rightarrow\ AP_{\beta_0=0.1} &= \tfrac{0.1}{x}+1+0.1x-0.01x^2 \\
Q_{\beta_0=1} &= 1+x+0.1x^2-0.01x^3 & \rightarrow\ AP_{\beta_0=1} &= \tfrac{1}{x}+1+0.1x-0.01x^2 \\
Q_{\beta_0=-1} &= -1+x+0.1x^2-0.01x^3 & \rightarrow\ AP_{\beta_0=-1} &= -\tfrac{1}{x}+1+0.1x-0.01x^2
\end{aligned}
$$

Now, for each function let's consider the values at which average product is maximized and how that affects the start of Stage II.

β_0	Value of x where AP_{max}	Start of Stage II	AP at Stage II
0	5	$x = 5$	1.25
0.1	(1.14, 4.78)	$x = 5$	1.25
1	Never	$x = 5$	1.25
-1	6.738	$x = 6.27$	1.074

The takeaways are as follows.

- When a production function has a positive intercept, the beginning of Stage II is determined using an assumed production function that does not have an intercept.
- When a production function has a negative intercept, the beginning of Stage II must explicitly account for the negative intercept.

6.2.3 Profit Maximization: Value-side Approach

We have so far examined optimal production strategies based only on the production function. However, in a market scenario, managerial economists will more likely be asked to consider optimal production strategies that are based on profit maximization.

First, we will consider the decision-making process from the revenue (or value) side. That is, assuming that firms are price takers and will sell their products at some given market price, P, decisions will be made based on the input–output relationship—a relationship that *can* be controlled or altered by the firm. Consider the following terms.

Total value product: TVP$= PQ$

Average value product: AVP= $\text{TVP}/x = \frac{PQ}{x} = P \cdot \text{AP}$

Marginal value product: MVP= $\frac{d\text{TVP}}{dx} = \frac{d(PQ)}{dx} = P\left(\frac{dQ}{dx}\right) = P \cdot \text{MP}$

These terms provide two notable insights. First, because prices, P, are simply accepted as is (i.e., as a constant) by firms, the shapes of the TVP, AVP, and MVP are analogous to the shapes of the TP, AP, and MP. Second, these terms show that the decision variable is x; that is, firms are maximizing profits by deciding how much inputs to use in order to produce some optimal amount of outputs.

Example 6.2.33 Determining the TVP, AVP, and MVP

Consider the production function $Q = x + 6x^{0.5} - 2x^{1.5}$. Determine the TVP, AVP, and MVP under the assumed market output price P.

TVP = $P(x + 6x^{0.5} - 2x^{1.5})$

AVP = $P + 6Px^{-0.5} - 2Px^{0.5}$

MVP = $P + 3Px^{-0.5} - 3Px^{0.5}$

Now let's consider the profit-maximization conditions in the value-side approach. A simple profit function is characterized by the equation

$$\pi = PQ - wx$$

where w is the price of inputs. Again, because firms are assumed to be price takers, w is treated as a constant that cannot be altered by any single firm.

The objective is to maximize profits by optimally choosing a level of inputs, x. We can, therefore, use the standard approach of differentiating π with respect to the choice variable, setting that first-order condition equal to zero, and then solving for x^*.

$$\frac{d\pi}{dx} = P\left(\frac{dQ}{dx}\right) - w = 0$$

This implies the profit-maximizing condition

$$\text{MVP} = w \quad \text{or} \quad \text{MP} = \frac{w}{P}$$

Intuitively, the first equality shows that profits are maximizing at the quantity of inputs such that the returns from using an additional unit of input are equal to the cost of the input unit. The second equality restates this condition in terms of price ratios. That is, profit maximization occurs at the point of production when an additional unit of output is equal to the ratio of input and output prices.

Profit Maximizing Condition in a Familiar Form: another way to restate the profit maximizing condition MVP $= w$ is as follows.

$$\begin{aligned} P\left(\frac{dQ}{dx}\right) &= w \\ P \cdot dQ &= w \cdot dx \\ \text{MR} &= \text{MC} \end{aligned}$$

The profit maximizing condition that marginal revenues must equal marginal costs is a fundamental property in economics.

After setting up the profit-maximizing condition, you can then solve for the optimal level of inputs that your firm will require, x^*, and the profit-maximizing level of output that the firm will supply, Q^*. We will work with a general case in which output and input prices, P and w, are unknown. Therefore, both x^* and Q^* will be functions of P and w. As a result, we will use the terminology ***input demand*** and ***supply*** functions to represent x^* and Q^*, respectively.

6.2.4 Profit Maximization: Cost-side Approach

In the value-side approach, a firm's objective is to choose a level of inputs that will maximize outputs/revenues subject to some cost of inputs. Therefore, the choice variable was x.

Another, analogous approach that firms can take is to search for profit-maximization by minimizing costs. That is, after determining the total costs of producing some output Q, firms seek to choose an optimal output level, Q^*, such that the costs to provide that output are minimized.

First, consider the following terms.

Variable costs (VC): costs that are affected by current decisions (e.g., level of output)

Fixed costs (FC): costs that are unaffected by current decisions

Total costs: TC= VC + FC

Average total costs: ATC= TC$/Q$

Average variable costs: AVC= VC$/Q$

Average fixed costs: AFC= FC$/Q$

Marginal costs: MC= $\frac{d\text{TC}}{dQ} = \frac{d\text{VC}}{dQ}$

There are several notable general properties associated with cost functions.

- AFC declines with greater output, Q.
- AVC is often convex.
- On the portion of the AVC curve that is positively sloped, MC will also be positively sloped.
- If AVC is convex such that there is a minimum, then AVC=MC at the point where AVC is minimized.

Now let's consider the profit-maximization conditions in the cost-side approach. Again, a simple profit function is characterized by the equation

$$\begin{aligned}\pi &= PQ - \text{VC} - \text{FC}\\ \pi &= PQ - wx - \text{FC}\end{aligned}$$

The objective here is to maximize profits by optimally choosing a cost-minimizing level of outputs, Q. We can, therefore, use the standard approach of differentiating π with respect to the choice variable, setting that first-order condition equal to zero, and then solving for Q^*.

$$\frac{d\pi}{dQ} = P - \frac{d\text{VC}}{dQ} = 0$$

This implies the profit-maximizing condition

$$P = \text{MC}$$

While it is not immediately apparent, the profit-maximizing conditions from the cost-side and from the value-side approaches are identical (which should intuitively make sense). We can see this by further decomposing the MC variable.

$$\begin{aligned}P &= \text{MC}\\ &= \tfrac{d(wx)}{dQ}\\ &= w\left(\tfrac{dx}{dQ}\right)\\ &= \tfrac{w}{dQ/dx}\\ P &= \tfrac{w}{\text{MP}}\\ P\cdot\text{MP} &= w\end{aligned}$$

After setting up the profit-maximizing condition, you can then solve for the optimal level of output at which the firm will minimize costs, Q^*, and the associated profit-maximizing level of inputs, x^*. As in the value-side approach, these will generally be characterized in terms of unknown output and input prices, P and w. Therefore, both Q^* and x^* will be functions of P and w.

6.2.5 Properties of the Input Demand and Supply Functions

Both the input demand and the supply functions are conditional on the level of output prices, P, and input prices, w. If one or both of these prices change, so will the profit-maximizing level of input demand and resulting level of a firm's output. These adjustments can be characterized by taking the first-order conditions of the input demand and supply functions with respect to P and x.

In general,

$$\frac{dQ(P,w)}{dP} > 0$$
$$\frac{dQ(P,w)}{dw} < 0$$

$$\frac{dx(P,w)}{dP} > 0$$
$$\frac{dx(P,w)}{dw} < 0$$

When output (market) prices are higher, the demand for inputs will increase as will total supply of a good or service. Conversely, if input prices (i.e., costs) are higher, the demand for those inputs will decrease as will the total supply of a good or service.

6.2.6 Indirect Profit Function

The input demand supply functions are useful because we can express them in terms of market output and input prices. Consequently, any time uncontrollable market conditions change, firms can immediately re-optimize their input demand and output supply to continue maximizing profits under the new market conditions.

Why not also have the same approach with profits? That is, why not develop a profit function that is only a function of uncontrollable market conditions, but that can be used to determine changes to profits if one or both of those conditions change?

We can derive this type of function using the input demand and supply functions. That is, after deriving $Q^*(P,w)$ and $x^*(P,w)$, we can substitute those values into the profit function to determine the ***indirect profit function***.

$$\pi(P,w) = P \cdot Q^*(P,w) - w \cdot x^*(P,w)$$

The indirect profit function is conditional only on output and input prices, and changes to firm's profits from altering market conditions can now be directly determined.

Hotelling's Lemma: a useful property of the indirect profit function is that we can directly derive the input demand and supply functions. This is the underlying idea of Hotelling's Lemma.

$$\frac{d\pi(P,w)}{dP} = Q^*(P,w)$$
$$\frac{d\pi(P,w)}{dw} = x^*(P,w)$$

6.2.7 Effects of Large Market Changes

Another convenient aspect of the input demand, supply, and indirect profit functions is that we can analyze how large market conditions (e.g., macroeconomic conditions) impact the choice of inputs and the resulting supply of outputs and profits. For example, what would happen if due to inflationary pressures, all prices (both inputs and outputs) increased by some scalar, α? That is, what if $P_{\text{new}} = \alpha \cdot P_{\text{old}}$ and $w_{\text{new}} = \alpha \cdot w_{\text{old}}$?

We can determine the changes by examining the homogeneity of a function. In general, a function $y = f(x,z)$ is homogeneous of degree k if $f(\alpha \cdot x, \alpha \cdot z) = \alpha^k f(x,z)$.

Example 6.2.34 Determining the Homogeneity of a Function

1. Determine the degree of homogeneity for the function, $Q(x) = x^{0.5}$.

$$\begin{aligned} Q(\alpha \cdot x) &= (\alpha \cdot x)^{0.5} \\ &= \alpha^{0.5} x^{0.5} \\ Q(\alpha \cdot x) &= \alpha^{0.5} Q(x) \end{aligned}$$

   In this case, $k = 0.5$. Therefore, the function $Q(x)$ is homogeneous of degree 0.5, (HOD = 0.5).

2. Determine the degree of homogeneity for the function, $Q(x,z) = x + z$.

$$\begin{aligned} Q(\alpha \cdot x, \alpha \cdot z) &= \alpha \cdot x + \alpha \cdot z \\ &= \alpha(x+z) \\ Q(\alpha \cdot x, \alpha \cdot z) &= \alpha^1 Q(x,z) \end{aligned}$$

In this case, $k = 1$. Therefore, the function $Q(x,z)$ is homogeneous of degree 1, (HOD = 1).

The degree of homogeneity provides insights about what will happen to the value of the function if all of the parameters of the function are scaled by some constant. Typically,

- The input demand function, $x^*(P,w)$, is HOD = 0. That is, scaling both output and input prices by some factor α does not change a firm's demand for inputs.
- The supply function, $Q^*(P,w)$, is HOD = 0. That is, scaling both output and input prices by some factor α does not change a firm's output.
- The indirect profit function, $\pi(P,w)$ is HOD = 1. That is, scaling both output and input prices by some factor α will scale the firm's profits by the same amount.

Example 6.2.35 Putting It All Together

Consider the production function $Q = 2x^{0.25}$. Provide an economic analysis of this function using both the value-side and the cost-side approaches.

Value-side			Cost-side		
TP	=	$Q = 2x^{0.25}$	x	=	$\frac{Q^4}{16}$
AP	=	$\frac{Q}{x} = 2x^{-0.75}$	VC	=	$wx = \frac{wQ^4}{16}$
MP	=	$\frac{dQ}{dx} = 0.5x^{-0.75}$	FC	=	0
Stage I:		None	ATC	=	$\frac{\text{TC}}{Q} = \frac{wQ^3}{16}$
Stage II:		$(x = 0, Q = 0)$ to $(x = \infty, Q = \infty)$	AVC	=	$\frac{\text{VC}}{Q} = \frac{wQ^3}{16}$
Stage III:		None	AFC	=	$\frac{\text{FC}}{Q} = 0$
TVP	=	$PQ = 2Px^{0.25}$	MC	=	$\frac{d\text{VC}}{dQ} = \frac{wQ^3}{4}$
AVP	=	$PQ/x = 2Px^{-0.75}$			
MVP	=	$\frac{d(PQ)}{x} = 0.5Px^{-0.75}$			
π_{max} cond:		MVP$= w$	π_{max} cond:		$P =$MC
		$0.5Px^{-0.75} = w$	$P = \frac{wQ^3}{4}$		
$x^*(P,w)$	=	$\left(\frac{P}{2w}\right)^{4/3}$	$Q^*(P,w)$	=	$\left(\frac{4P}{w}\right)^{1/3}$
$Q^*(P,w)$	=	$\left(\frac{4P}{w}\right)^{1/3}$	$x^*(P,w)$	=	$\left(\frac{P}{2w}\right)^{4/3}$
$\pi(P,w)$	=	$1.2\left(\frac{P^{4/3}}{w^{1/3}}\right)$	$\pi(P,w)$	=	$1.2\left(\frac{P^{4/3}}{w^{1/3}}\right)$

$$\text{HOD}_{\text{supply}}: Q(\alpha P, \alpha w) = \left(\frac{4\alpha P}{\alpha w}\right)^{1/3} = \alpha^0 \left(\frac{4P}{w}\right)^{1/3}$$

$$\text{HOD}_{\text{input}}: x(\alpha P, \alpha w) = \left(\frac{\alpha P}{2\alpha w}\right)^{4/3} = \alpha^0 \left(\frac{P}{2w}\right)^{4/3}$$

$$\text{HOD}_{\text{profit}}: pi(\alpha P, \alpha w) = 1.2\left(\frac{(\alpha P)^{4/3}}{(\alpha w)^{1/3}}\right) = \alpha^1 \cdot 1.2\left(\frac{P^{4/3}}{w^{1/3}}\right)$$

Practice Problems

1. Determine the three production stages for the following production functions.
   (a) $Q = x + 0.25x^2 - 0.001x^3$
   (b) $Q = x^2 - 0.025x^3$
   (c) $Q = -1 + x^2 - 0.025x^3$
2. Consider the production function $Q = x - 0.1x^2$ and suppose that $P = 4$ and $w = 2$. Solve for the profit-maximizing level of inputs and outputs and the resulting profit.
3. Determine the degree of homogeneity for the following functions.
   (a) $y(x,z) = \frac{x^2}{z}$
   (b) $y(x,z) = \frac{x}{x+z}$
4. For each of the following production functions, solve for the input demand, supply, and indirect profit functions using both the value-side and the cost-side approaches. Then determine the degree of homogeneity for each of the three functions.
   (a) $Q = x^{0.25}$
   (b) $Q = x^{0.25}$ and FC$= 10$.
   (c) $Q = 10 + x^{0.5}$

6.3 Uncertainty and Production

We have so far only assessed production theory under the assumption of certainty. This is convenient but, of course, not particularly realistic. For example, a firm is never certain what the level of sales will be or the price of inputs in any particular year.

Our strategy, therefore, for incorporating the concept of uncertainty into production is going to be this:

1. Determine the optimal profit-maximizing production input functions, Q^* and x^*.
2. Assess the expected value of profit using a probabilistic model.

That is, first we will determine the underlying optimal profit function of only market prices, P, and input prices, w, which are assumed to be known.

$$\pi^* = f(P, w_1, w_2, \ldots)$$

Then, relax the assumption about known prices and calculate the expected profit function.

$$E[\pi^*] = f(E[P], E[w_1], E[w_2], \ldots)$$

Example 6.3.36 Determining Expected Profits

Suppose that you estimate the optimal profit function,

$$\pi^* = 2P^{0.5}w_1^{0.5}w_2$$

You also know there are two economic markets scenarios with different probabilities of occurrence.

Market Scenario	**Prob of occurring**	P	w_1	w_2
Good	60%	\$5.00	\$2.00	\$4.00
Bad	40%	\$2.00	\$3.00	\$5.00

What is the expected profit for the firm?

Using the formula $E[\pi^*] = f(E[P], E[w_1], E[w_2], \ldots)$, we see that we need to first determine $E[P]$, $E[w_1]$, and $E[w_2]$. These are:

$$\begin{aligned} E[P] &= (0.6) \times (\$5.00) + (0.40) \times (\$2.00) \\ &= \$3.80 \end{aligned}$$

$$\begin{aligned} E[w_1] &= (0.6) \times (\$2.00) + (0.40) \times (\$3.00) \\ &= \$2.40 \end{aligned}$$

$$\begin{aligned} E[w_2] &= (0.6) \times (\$4.00) + (0.40) \times (\$5.00) \\ &= \$4.40 \end{aligned}$$

Now we can calculate the expected optimal profit as

$$
\begin{aligned}
E[\pi^*] &= 2 \times (\$3.80)^{0.5} \times (\$2.40)^{0.5} \times (\$4.40) \\
&= \$26.57
\end{aligned}
$$

Appendix 1: Excel Setup for Empirical Analysis

Setting up the Analysis ToolPak in Excel

Microsoft Excel is a widely available spreadsheet software that has useful capabilities for analyzing data. Many of these capabilities are provided through the "Analysis Toolpak" add-in. The following are step-by-step directions for installing this add-in for Excel 2013. For Excel 2010, the installation procedures are similar.[1] For Mac machines, you will need to download a free third-party software, StatPlus:mac LE (AnalystSoft Inc.) from the following URL: http://www.analystsoft.com/en/products/statplusmacle/download.phtml.

1. Open Excel 2013.

2. Click the "File" tab and select "Options." The *Excel Options* window will open.

3. In the left selection pane, click "Add-Ins." The *Excel Options* will now display the active and inactive application add-ins. If the "Analysis ToolPak" is listed in the *Inactive Application Add-ins* list, proceed to the next step. Otherwise, if the "Analysis ToolPak" is listed in the *Active Application Add-ins*, close the *Excel Options* window and proceed to step 6.

4. At the bottom of the *Excel Options* window, ensure that the *Manage* drop-down menu displays "Excel Add-ins." Then, click the "Go…" button. The *Add-Ins* window will open.

5. In the *Add-Ins available* list, select the "Analysis ToolPak" by click the associated checkbox. Then, click the "OK" button. The window will close.

6. Click the "Data" tab. On the right-hand side of the tab, there will be a "Data Analysis" button.

[1] For detailed directions, see http://office.microsoft.com/en-us/excel-help/load-the-analysis-toolpak-HP010021569.aspx.

Descriptive Statistics in the Analysis ToolPak

The Excel Analysis ToolPak provides a number of useful statistical tools for generating descriptive statistics for a data set. Using the Analysis ToolPak for creating descriptive statistics is relatively straightforward and your experience with Excel should be sufficient to be successful in generating this information. For this course, the following Analysis ToolPak options will be of most use:

- Descriptive Statistics
- Correlation
- Covariance
- Histogram
- Regression

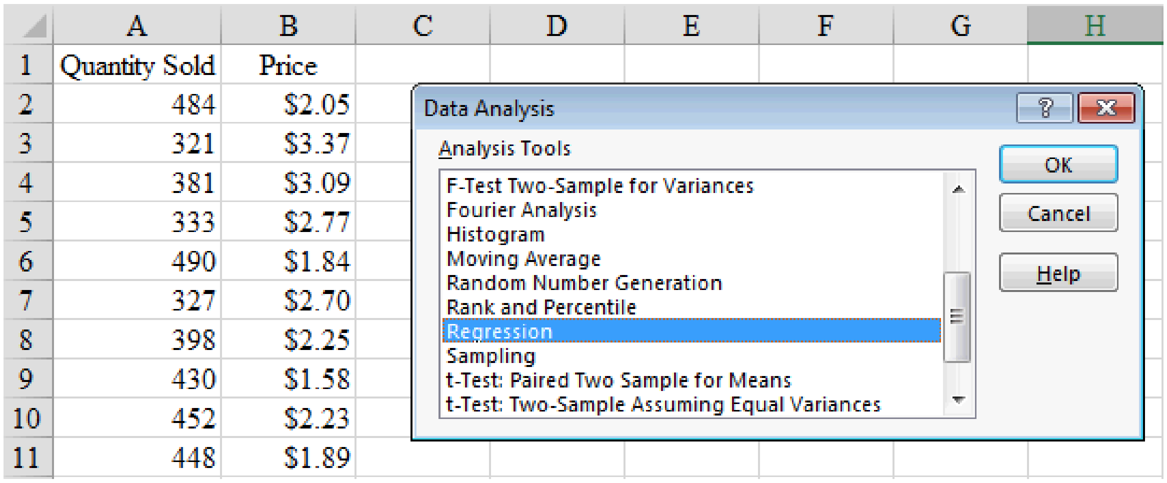
Using Excel for Regression Analysis

The following example provides step-by-step instructions for performing regression analysis using the Analysis ToolPak. The example assumes a single explanatory X variable, but the same steps can be used to perform multiple regressor analyses.

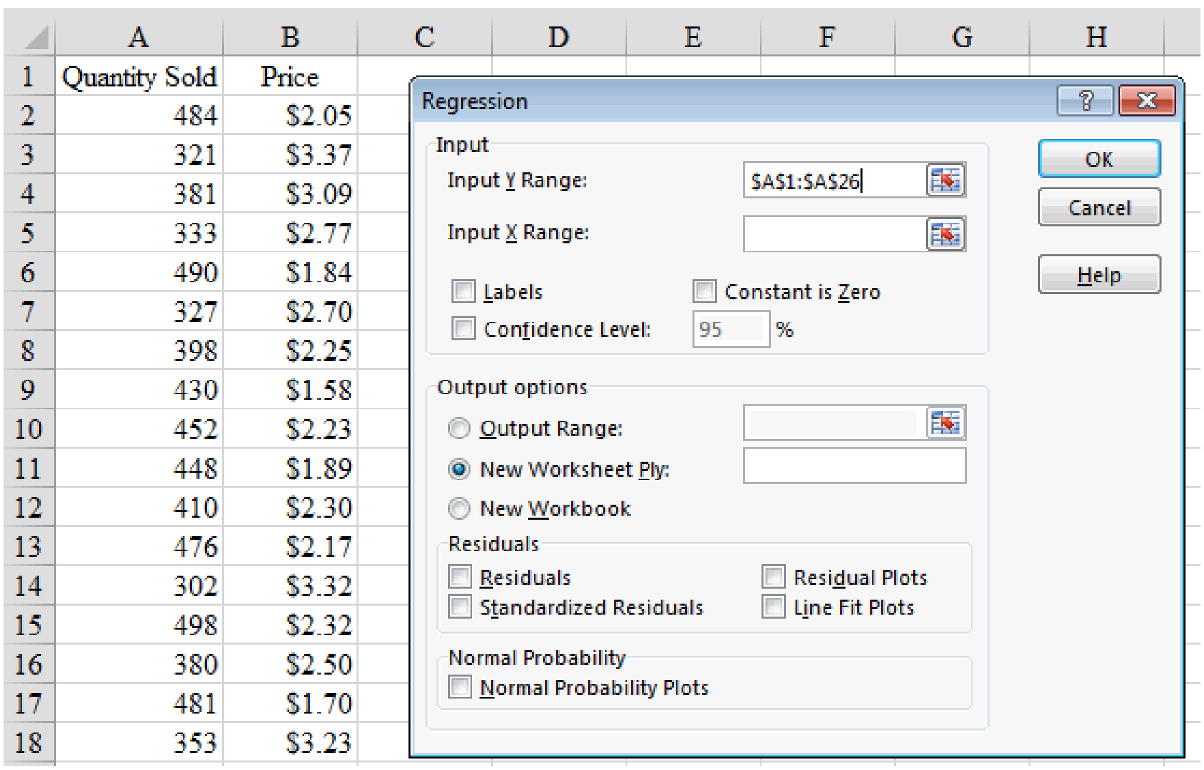
1. Consider a data set that has a dependent variable, Y, and an independent explanatory variable X. For example, the dependent variable might be the quantity of products sold and the explanatory variable is price.

	A	B	C	D	E
1	Quantity Sold	Price			
2	484	$2.05			
3	321	$3.37			
4	381	$3.09			
5	333	$2.77			
6	490	$1.84			
7	327	$2.70			
8	398	$2.25			
9	430	$1.58			

2. When you open the initial Analysis ToolPak window, scroll down and double-click the "Regression" option.

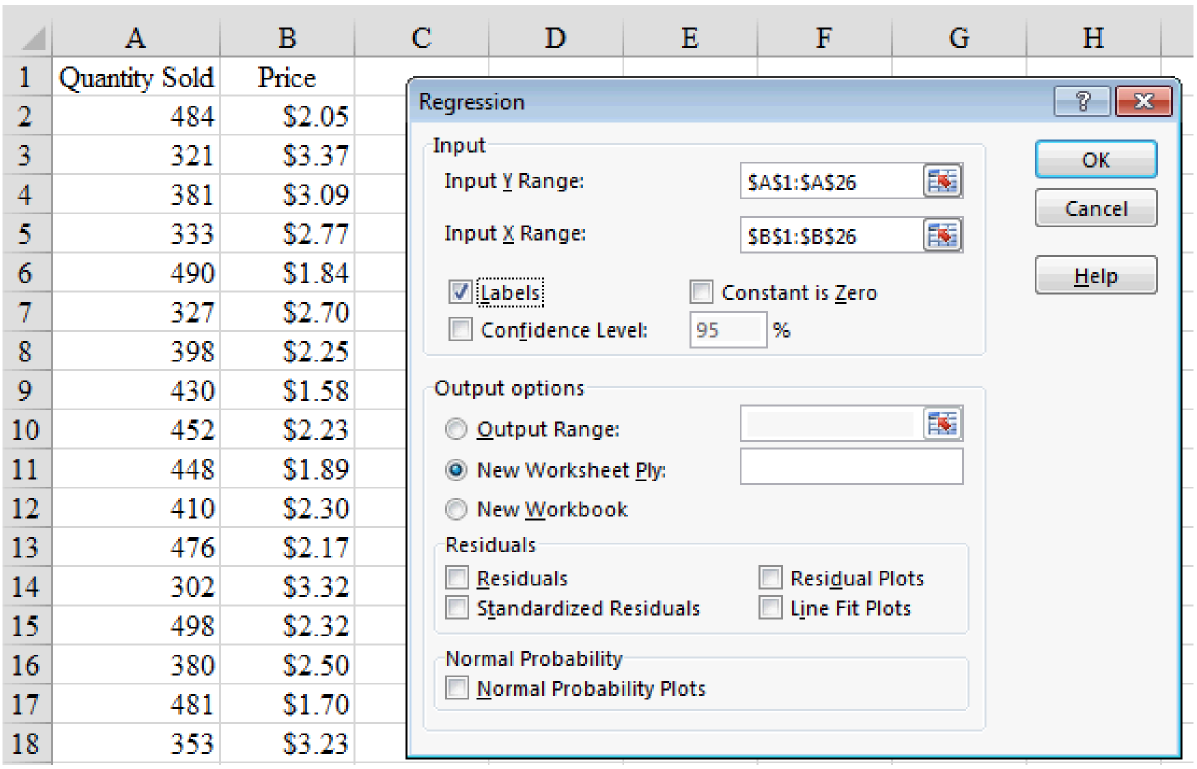


3. Click inside the "Input Y Range" selection area and select the spreadsheet cells containing data for the dependent variable. Include the row heading in the selection.



4. Click inside the "Input X Range" selection area and select the spreadsheet cells containing data for the independent explanatory variable. Include the row heading in the selection. If there are more than one explanatory variables, select all of the cells in the columns and rows corresponding to those data.

5. Click the "Labels" checkbox in the *Regression* window.



6. Click the “OK” button. A new Excel spreadsheet will be created displaying the regression results.

	A	B	C	D	E	F	G	H	I
1	SUMMARY OUTPUT								
2									
3	*Regression Statistics*								
4	Multiple R	0.749288615							
5	R Square	0.561433429							
6	Adjusted R Square	0.542365317							
7	Standard Error	45.66591351							
8	Observations	25							
9									
10	ANOVA								
11		*df*	*SS*	*MS*	*F*	*Significance F*			
12	Regression	1	61400.9199	61400.92	29.44358	1.63081E-05			
13	Residual	23	47963.6401	2085.376					
14	Total	24	109364.56						
15									
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
17	Intercept	623.1666523	42.53264562	14.65149	3.74E-13	535.1811712	711.15213	535.181171	711.152133
18	Price	-91.1006498	16.78905243	-5.42619	1.63E-05	-125.8314509	-56.369849	-125.831451	-56.3698487
19									

Appendix 2: Statistical Tables

Table A1.1: Cumulative Areas Under the Standard Normal Distribution

z	0	1	2	3	4	5	6	7	8	9
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121

(continued on next page. . .)

z	0	1	2	3	4	5	6	7	8	9
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Example: If $z \sim N(0,1)$, then $P[Z \leq -1.45] = 0.0735$. Similarly, $P[Z \leq 2.38] = 0.9913]$.

Table A1.2: Critical Values of the t Distribution

		Critical value, α				
	1-tailed	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**
	2-tailed	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**
Degrees of Freedom	**1**	3.078	6.314	12.706	31.821	63.657
	2	1.886	2.920	4.303	6.965	9.925
	3	1.638	2.353	3.182	4.541	5.841
	4	1.533	2.132	2.776	3.747	4.604
	5	1.476	2.015	2.571	3.365	4.032
	6	1.440	1.943	2.447	3.143	3.707
	7	1.415	1.895	2.365	2.998	3.499
	8	1.397	1.860	2.306	2.896	3.355
	9	1.383	1.833	2.262	2.821	3.250
	10	1.372	1.812	2.228	2.764	3.169
	11	1.363	1.796	2.201	2.718	3.106
	12	1.356	1.782	2.179	2.681	3.055
	13	1.350	1.771	2.160	2.650	3.012
	14	1.345	1.761	2.145	2.624	2.977
	15	1.341	1.753	2.131	2.602	2.947
	16	1.337	1.746	2.120	2.583	2.921
	17	1.333	1.740	2.110	2.567	2.898
	18	1.330	1.734	2.101	2.552	2.878
	19	1.328	1.729	2.093	2.539	2.861
	20	1.325	1.725	2.086	2.528	2.845
	21	1.323	1.721	2.080	2.518	2.831
	22	1.321	1.717	2.074	2.508	2.819
	23	1.319	1.714	2.069	2.500	2.807
	24	1.318	1.711	2.064	2.492	2.797
	25	1.316	1.708	2.060	2.485	2.787
	26	1.315	1.706	2.056	2.479	2.779
	27	1.314	1.703	2.052	2.473	2.771
	28	1.313	1.701	2.048	2.467	2.763
	29	1.311	1.699	2.045	2.462	2.756
	30	1.310	1.697	2.042	2.457	2.750
	40	1.303	1.684	2.021	2.423	2.704
	60	1.296	1.671	2.000	2.390	2.660
	90	1.291	1.662	1.987	2.368	2.632
	120	1.289	1.658	1.980	2.358	2.617
	∞	1.282	1.646	1.962	2.330	2.581

Example: for a 5% significance value two-tailed test with 20 df, the critical value is 2.086.

Table A1.3: 5% Critical Values of the F Distribution

Denominator Degrees of Freedom	Numerator Degrees of Freedom: 1	2	3	4	5	6	7	8	9	10
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83

Example: for a 5% significance value with numerator $df = 8$ and denominator $df = 40$, the critical value is 2.18.