ECNS 309, Fall 2014

Practice Problems #4

1. Market versus non-market differences.

   Non-market
   Evaluate utility and production at the tangency, MRPS = MRS. That is, set $\frac{dQ_1}{dQ_2} = \frac{dQ_1}{dQ_2}$ and solve for Q_1 and Q_2.

$$\begin{aligned} \text{MRPS} &= -2Q_2 \\ \text{MRS} &= -\tfrac{2}{3}U^2/Q_2^{5/3} = -\tfrac{2}{3}(Q_1^{1/2}Q_2^{1/3})^2/Q_2^{5/3} = -\tfrac{2}{3}\tfrac{Q_1}{Q_2} \end{aligned}$$

$$\begin{aligned} Q_1 &= 7.5 \\ Q_2 &= 1.58 \\ &\quad \text{(If prices were as in the market case, } \pi = 16.58\text{)} \\ U &= 3.19 \end{aligned}$$

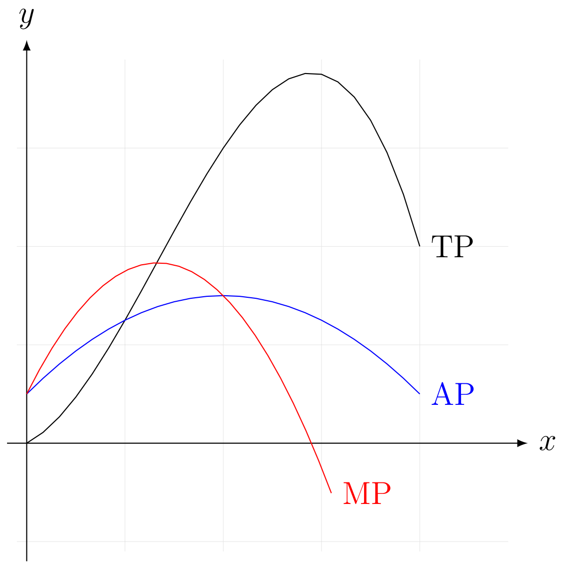
   Market

$$\begin{aligned} \text{MRPS} &= \tfrac{dQ_1}{dQ_2} = -\tfrac{P_2}{P_1} \\ &= -0.5 = -2Q_2 \\ Q_2 &= 0.25 \\ Q_1 &= 9.93 \\ \pi &= 20.11 \end{aligned}$$

$$\begin{aligned} \text{MRS} &= \tfrac{dQ_1}{dQ_2} = -\tfrac{P_2}{P_1} \\ Q_2 &= \tfrac{2}{3}Q_1\tfrac{P_1}{P_2} \\ &\quad \text{Plug into the profit function} \\ Q_1 &= \tfrac{3}{5}\tfrac{\pi}{P_1} = \tfrac{3}{5}\tfrac{20.11}{2} = 6.03 \\ Q_2 &= 8.04 \\ U &= 4.92 \end{aligned}$$

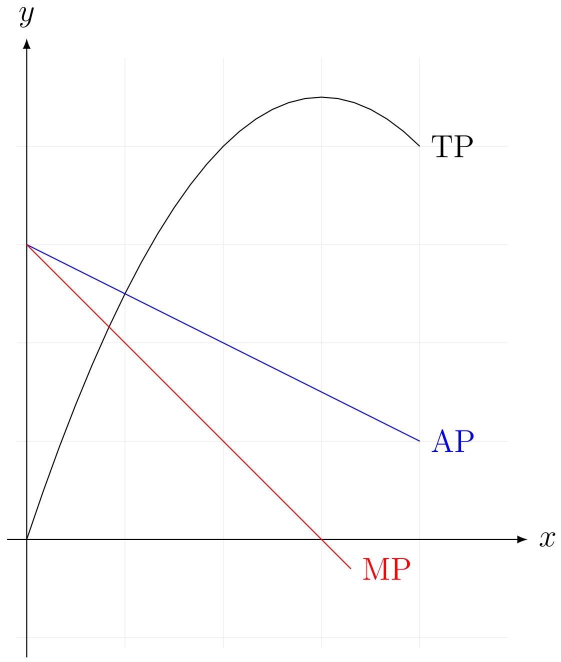
2. For each production function, (i) graph the total, average, and marginal product functions and (ii) identify the three production stages.

(a) $Q = 0.5x + x^2 - 0.25x^3$



$$\begin{array}{llll} \text{Stage I}: & (Q = 0, x = 0) & \text{to} & (Q = 3, x = 2) \\ \text{Stage II}: & (Q = 3, x = 2) & \text{to} & (Q = 3.76, x = 2.90) \\ \text{Stage III}: & (Q > 3.76) & & \end{array}$$

(b) $Q = 3x - 0.5x^2$

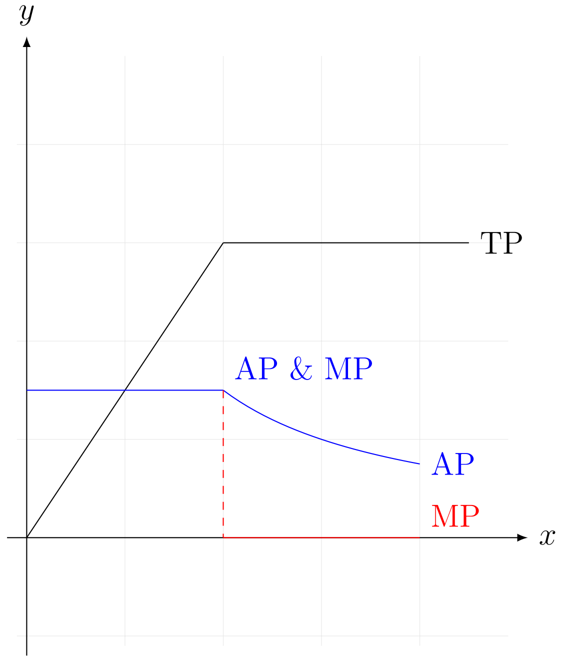


Stage I : None

Stage II : $(Q = 0, x = 0)$ to $(Q = 4.5, x = 3)$

Stage III : $(Q > 4.5)$

(c) $Q = \begin{cases} 1.5x, & \text{if } 0 \leq x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$



Stage I :	None
Stage II :	$(Q = 0, x = 0)$ to $(Q = 3, x = 2)$
Stage III :	$(Q > 3)$

3. For each production function, determine the input demand function.

   (a) $Q = 2x - x^2$

$$x(P, w) = \frac{2P - w}{2P}$$

   (b) $Q = x + 0.2x^2 - x^3$

$$x(P, w) = \frac{-0.4 \pm \sqrt{0.16 + 12(1 - \frac{w}{P})}}{-6}$$

4. Consider the following production function, output price, and input price combinations. Solve for the profit-maximizing input and output amounts as well as the revenue, variable cost, and profit values. Then, graph the total, average, and marginal value products and the revenues, costs, and profits.

   (a) $Q = 3x - 0.25x^2$, $P = 5$, $w = 2$.

$$\begin{aligned} x^* &= 5.2 \\ Q^* &= 8.84 \\ \text{Revenue} &= 44.2 \\ \text{VC} &= 10.4 \\ \pi^* &= 33.8 \end{aligned}$$

   (b) $Q = 0.5x + x^2 - 0.1x^3$, $P = 3$, $w = 1$.

$$\begin{aligned} x^* &= 6.75 \\ Q^* &= 18.18 \\ \text{Revenue} &= 54.54 \\ \text{VC} &= 6.75 \\ \pi^* &= 47.79 \end{aligned}$$

   (c) $Q = 1 + x - 2x^2$, $P = 4$, $w = 1$.

$$\begin{aligned} x^* &= 0.19 \\ Q^* &= 1.11 \\ \text{Revenue} &= 4.44 \\ \text{VC} &= 0.19 \\ \pi^* &= 4.25 \end{aligned}$$

5. Discuss the mathematical and economic intuition of when and why optimal production strategies change when $Q \neq 0$ when $x = 0$.

   Answers will vary.

6. Determine the degree of homogeneity for the following production functions. Interpret.

   (a) HOD=1. If inputs, x and z, double, then outputs will double.

   (b) HOD=-1. If inputs, x and z, double, then outputs will decrease by double.

   (c) HOD=0.5. If we double fertilizer and rain amounts, outputs will increase by 50%.

7. Provide an economic description for the following production function using the production-side and cost-side approaches. When determining the cost-side approach, determine the homogeneity of the supply, input demand, and profit functions.

(a) $Q = 3 + x^{0.5}$

Production-side			Cost-side		
TP	$=$	$3 + x^{0.5}$	TC	$=$	$w(Q-3)^2$
AP	$=$	$\frac{3}{x} + x^{-0.5}$	VC	$=$	$w(Q-3)^2$
MP	$=$	$0.5x^{-0.5}$	FC	$=$	0
TVP	$=$	$3P + Px^{0.5}$	ATC	$=$	$wQ - 6w + 9w/Q$
AVP	$=$	$P(\frac{3}{x} + x^{-0.5})$	AVC	$=$	$wQ - 6w + 9w/Q$
MVP	$=$	$0.5Px^{-0.5}$	AFC	$=$	0
			MC	$=$	$2w(Q-3)$
Stage 1:		None			
Stage 2:		$(Q = 0, x = 0)$ to $(Q = \infty, x = \infty)$			
Stage 3:		None			
$\pi_{\max}$ when:		$w = 0.5Px^{-0.5}$	$\pi_{\max}$ when:		$P = 2w(Q-3)$
x^*	$=$	$\left(\frac{0.5P}{w}\right)^2$	Supply, Q	$=$	$\left(\frac{3w+0.5P}{w}\right)$
Q^*	$=$	$\left(\frac{3w+0.5P}{w}\right)$	Input demand, x	$=$	$\left(\frac{0.5P}{w}\right)^2$
			IPC, π	$=$	$\frac{P^2+12Pw}{4w}$

(b) $Q = x^{1/3}/2$ and $FC = 2$

Production-side			Cost-side		
TP	$=$	$x^{1/3}/2$	TC	$=$	$8wQ^3 + 2$
AP	$=$	$x^{-2/3}/2$	VC	$=$	$8wQ^3$
MP	$=$	$\frac{1}{6}x^{-2/3}$	FC	$=$	2
TVP	$=$	$Px^{1/3}/2$	ATC	$=$	$8wQ^2 + 2/Q$
AVP	$=$	$Px^{-2/3}/2$	AVC	$=$	$8wQ^2$
MVP	$=$	$\frac{P}{6}x^{-2/3}$	AFC	$=$	$2/Q$
			MC	$=$	$24wQ^3$
Stage 1:		None			
Stage 2:		$(Q = 0, x = 0)$ to $(Q = \infty, x = \infty)$			
Stage 3:		None			
$\pi_{\max}$ when:		$w = \frac{P}{6}x^{-2/3}$	$\pi_{\max}$ when:		$P = 24wQ^3$
x^*	$=$	$\left(\frac{P}{6w}\right)^{1.5}$	Supply, Q	$=$	$\left(\frac{P}{24w}\right)^{0.5}$
Q^*	$=$	$\left(\frac{P}{24w}\right)^{0.5}$	Input demand, x	$=$	$\left(\frac{P}{6w}\right)^{1.5}$
			IPC, π	$=$	$\frac{0.18P^{1.5}}{w^{0.5}} - 2$

(c) $Q = (x-2)^{0.8}$

Production-side			Cost-side		
TP	$=$	$(x-2)^{0.8}$	TC	$=$	$w(2+Q^{1.25})$
AP	$=$	$\frac{(x-2)^{0.8}}{x}$	VC	$=$	$w(2+Q^{1.25})$
MP	$=$	$0.8(x-2)^{-0.2}$	FC	$=$	0
TVP	$=$	$P(x-2)^{0.8}$	ATC	$=$	$\frac{2w}{Q} + wQ^{0.25}$
AVP	$=$	$\frac{P(x-2)^{0.8}}{x}$	AVC	$=$	$\frac{2w}{Q} + wQ^{0.25}$
MVP	$=$	$0.8P(x-2)^{-0.2}$	AFC	$=$	0
			MC	$=$	$1.25wQ^{0.25}$
Stage 1:		None			
Stage 2:		$(Q=0, x=0)$ to $(Q=\infty, x=\infty)$			
Stage 3:		None			
$\pi_{\max}$ when:		$w = 0.8P(x-2)^{-0.2}$	$\pi_{\max}$ when:		$P = 1.25wQ^{0.25}$
x^*	$=$	$2 + \left(\frac{P}{1.25w}\right)^5$	Supply, Q	$=$	$\left(\frac{P}{1.25w}\right)^4$
Q^*	$=$	$\left(\frac{P}{1.25w}\right)^4$	Input demand, x	$=$	$2 + \left(\frac{P}{1.25w}\right)^5$
			IPC, π	$=$	$\frac{0.8P^5}{w^4} - 2w$

8. Suppose that the production function is characterized as $Q = \alpha x^{\beta}$, where α and β are some constant parameters.

(a) Derive the supply, input demand, and indirect profit functions.

$$Q = \alpha\left(\frac{P\alpha\beta}{w}\right)^{\frac{\beta}{1-\beta}}$$

$$x = \left(\frac{P\alpha\beta}{w}\right)^{\frac{1}{1-\beta}}$$

$$\pi = w^{\frac{\beta}{1-\beta}} P^{\frac{1}{1-\beta}} \left[\alpha(\alpha\beta)^{\frac{\beta}{1-\beta}} - (\alpha\beta)^{\frac{1}{1-\beta}}\right]$$

(b) Now suppose that there is a technological improvement such that $\alpha = (\alpha+\theta)$, where θ represents a change in technology. Re-derive the supply, input demand, and indirect profit functions.

$$Q = (\alpha+\theta)\left(\frac{P(\alpha+\theta)\beta}{w}\right)^{\frac{\beta}{1-\beta}}$$

$$x = \left(\frac{P(\alpha+\theta)\beta}{w}\right)^{\frac{1}{1-\beta}}$$

$$\pi = w^{\frac{\beta}{1-\beta}} P^{\frac{1}{1-\beta}} \left[(\alpha+\theta)\{(\alpha+\theta)\beta\}^{\frac{\beta}{1-\beta}} - (\alpha\beta)^{\frac{1}{1-\beta}}\right]$$

(c) Discuss the effects on supply, input demand, and profit if (i) $\theta < 0$ and (ii) $\theta > 0$.

In all three cases, $\frac{dQ}{d\theta} = \frac{dx}{d\theta} = \frac{d\pi}{d\theta} > 0$. Therefore, as if technology improves, $\theta > 0$, supply, input demand, and profits increase. Conversely, if $\theta < 0$, all three aspects decrease.

9. The following are intermediate steps:

$$\begin{aligned}
x_1(x_2, w_1, w_2) &= 1.5x_2\tfrac{w_2}{w_1} \\
x_2(x_1, w_1, w_2) &= \tfrac{2}{3}x_1\tfrac{w_1}{w_2} \\
x_1(Q, w_1, w_2) &= 1.18Q^{1.2}\left(\tfrac{w_2}{w_1}\right)^{0.4} \\
x_2(Q, w_1, w_2) &= 0.78Q^{1.2}\left(\tfrac{w_1}{w_2}\right)^{0.6} \\
VC(Q, w_1, w_2) &= 1.96Q^{1.2}w_1^{0.6}w_2^{0.4} \\
MC &= 2.35Q^{0.2}w_1^{0.6}w_2^{0.4} \\
Q^*(P, w_1, w_2) &= \tfrac{P^5}{72w_1^3w_2^2}
\end{aligned}$$

10. The following are intermediate steps:

$$\begin{aligned}
x_1(x_2, w_1, w_2) &= x_2\tfrac{\alpha w_2}{\beta w_1} \\
x_2(x_1, w_1, w_2) &= x_1\tfrac{\beta w_1}{\alpha w_2} \\
x_1(Q, w_1, w_2) &= \left(\tfrac{\alpha w_2}{\beta w_1}\right)^{\frac{\beta}{\alpha+\beta}} Q^{\frac{1}{\alpha+\beta}} \\
x_2(Q, w_1, w_2) &= \left(\tfrac{\beta w_1}{\alpha w_2}\right)^{\frac{\alpha}{\alpha+\beta}} Q^{\frac{1}{\alpha+\beta}} \\
VC(Q, w_1, w_2) &= w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}} \left(\tfrac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \left(\tfrac{\alpha+\beta}{\alpha}\right) Q^{\frac{1}{\alpha+\beta}} \\
Q^*(P, w_1, w_2) &= \left\{P^{\alpha+\beta}\left(\tfrac{\alpha}{w_1}\right)^{\alpha}\left(\tfrac{\beta}{w_2}\right)^{\beta}\right\}^{\frac{1}{1-\alpha-\beta}} \\
x_1^*(P, w_1, w_2) &= \left\{P\left(\tfrac{\alpha}{w_1}\right)^{1-\beta}\left(\tfrac{\beta}{w_2}\right)^{\beta}\right\}^{\frac{1}{1-\alpha-\beta}} \\
x_2^*(P, w_1, w_2) &= \left\{P\left(\tfrac{\alpha}{w_1}\right)^{\alpha}\left(\tfrac{\beta}{w_2}\right)^{1-\alpha}\right\}^{\frac{1}{1-\alpha-\beta}} \\
\pi^*(P, w_1, w_2) &= \left\{Pw_1^{-\alpha}w_2^{-\beta}\alpha^{1-\beta}\beta^{1-\alpha}\right\}^{\frac{1}{1-\alpha-\beta}}
\end{aligned}$$

11. The following are intermediate steps:

$$
\begin{aligned}
x_1(x_2, w_1, w_2) &= 3x_2 \tfrac{w_2}{w_1} \\
x_2(x_1, w_1, w_2) &= 0.33x_1 \tfrac{w_1}{w_2} \\
x_1(Q, w_1, w_2) &= 1.32Q^{5/4} \left(\tfrac{w_1}{w_2}\right)^{0.25} \\
x_2(Q, w_1, w_2) &= 0.44Q^{5/4} \left(\tfrac{w_1}{w_2}\right)^{0.75} \\
VC(Q, w_1, w_2) &= 1.76Q^{5/4} w_1^{0.75} w_2^{0.25} \\
Q^*(P, w_1, w_2) &= \tfrac{0.43P^4}{w_1^3 w_2} \\
x_1^*(P, w_1, w_2) &= \tfrac{0.03P^5}{w_1^4 w_2} \\
x_2^*(P, w_1, w_2) &= \tfrac{0.01P^5}{w_1^3 w_2^2} \\
\pi^*(P, w_1, w_2) &= \tfrac{0.01P^5}{w_1^3 w_2}
\end{aligned}
$$