Review of Mathematical Concepts Used in Managerial Economics

Economics is the most mathematical of all the social sciences. Indeed, to the uninitiated reader, many academic journals in economics resemble a mathematics or physics journal. Because this text is intended to show the practical applications of economic theory, this presents something of a dilemma. On the one hand, the economic theory of managerial decision making has evolved along with the rest of economics to a point where it can be (and usually is) profusely expressed in mathematical terms. On the other hand, industry experience indicates that managers seldom use the more advanced mathematical expressions of economic theory. They do, nonetheless, rely quite often on many of the concepts, graphs, and relatively simple numerical examples that are used throughout this text to assist them in their decision making.

But the dilemma does not end here. Regardless of the role of mathematics in managerial decision making, it certainly serves as an important instructional vehicle for economics professors. Using calculus enables the very concise expression of complex functional relationships and the quick solution of problems involving the optimal allocation of scarce resources. Moreover, students with extensive academic backgrounds or work experience in applied mathematics (i.e., engineers and scientists) often find that they are able to discern the essential nature of an economic problem more easily with equations and calculus than with narratives and tabular examples.

We have resolved the dilemma in the following way. The explanations of economic terms, concepts, and methods of analysis rely primarily on verbal definitions, numerical tables, and graphs. As appropriate, chapter appendixes present the same material using algebra and calculus. At times, algebra and calculus are employed in the main body of a chapter. Moreover, problems and exercises at the end of the chapter give students ample opportunity to reinforce their understanding of the material with the use of algebra and calculus, as well as with tables and graphs.

The authors' experience as teachers indicates that many students have already learned the mathematics employed in this text, both in the main body and in the appendixes. However, some students may have studied this material some time ago and may therefore benefit from a review. Such a review is offered in the balance of this appendix. It is intended only as a brief refresher. For a more comprehensive review, readers should consult any of the many texts and review books on this subject.[1] In fact, any college algebra or calculus text would be just as suitable as a reference.

Variables, Functions, and Slopes: The Heart of Economic Analysis

A *variable* is any entity that can assume different values. Each academic discipline focuses attention on its own set of variables. For example, in the social sciences, political scientists may study power and authority, sociologists may study group cohesiveness, and psychologists may study paranoia. Economists study such variables as price, output, revenue, cost, and profit. The advantage that economics has over the other social sciences is that most of its variables can be measured in a relatively unambiguous manner.[2]

Once the variables of interest have been identified and measured, economists try to understand how and why the values of these variables change. They also try to determine what conditions will lead to optimal values. Here the term *optimal* refers to the best possible value in a particular situation. *Optimal* may refer to the maximum value (as in the case of profit), or it may refer to the minimum value (as in the case of cost). In any event, the analysis of the changes in a variable's value, often referred to as a variable's "behavior," is almost always carried out in relation to other variables. In mathematics, the relationship of one variable's value to the values of other variables is expressed in terms of a function. Formally stated in mathematical terms, Y is said to be a function of X (i.e., $Y = f(x)$, where f represents "function" if for any value that might be assigned to X a value of Y can be determined). For example, the demand function indicates the quantity of a good or service that people are willing to buy, given the values of price, tastes and preferences, prices of related products, number of buyers, and future expectations. A functional relationship can be expressed using tables, graphs, or algebraic equations.

To illustrate the different ways of expressing a function, let us use the total revenue function. Total revenue (TR, or sales) is defined as the unit price of a product (P) multiplied by the number of units sold (Q). That is, $\text{TR} = P \times Q$. In economics, the general functional relationship for total revenue is that its value depends on the number of units sold. That is, $\text{TR} = f(Q)$. Total revenue, or TR, is called the *dependent variable* because its value depends on the value of Q. Q is called the *independent variable* because its value may vary independently of the value of TR. For example, suppose a product is sold for $5 per unit. Table 1 shows the relationship between total revenue and quantity over a selected range of units sold.

Figure 1 shows a graph of the values in Table 1. As you can see in this figure, we have related total revenue to quantity in a linear fashion. There does not always have to be a linear relationship between total revenue and quantity. As you see in the next section, this function, as well as many other functions of interest to economists, may assume different nonlinear forms.

[1]See, for example, Bodh R. Gulati, *College Mathematics with Applications to the Business and Social Sciences*, New York: Harper & Row, 1978; and Donald and Mildred Stanel, *Applications of College Mathematics*, Lexington, MA: D.C. Heath, 1983.

[2]To be sure, variables in the other social sciences are measurable, but in many instances, the measurement standards themselves are subject to discussion and controversy. For example, psychologists may use the result of some type of IQ test as a measure of intelligence. But there is an ongoing debate as to whether this result is reflective of one's native intelligence or socioeconomic background.

Units Sold (Q)	Total Revenue (TR)
0	$0
1	5
2	10
3	15
4	20
5	25

Table 1 Tabular Expression of the TR Function

We can also express the relation depicted in Table 1 and Figure 1 in the following equation:

$$\mathrm{TR} = 5Q \tag{1}$$

where TR = Dependent variable (total revenue)
Q = Independent variable (quantity)
5 = Coefficient showing the relationship between changes in TR relative to changes in Q

This equation can also be expressed in the general form:

$$Y = a + bX \tag{2}$$

where Y = Dependent variable (i.e., TR)
X = Independent variable (i.e., Q)
b = Coefficient of X (i.e., the given P of $5)
a = Intercept term (in this case, 0)

In a linear equation, the coefficient b (which takes the value of 5 in our total revenue function) can also be thought of as the change in Y over the change in X

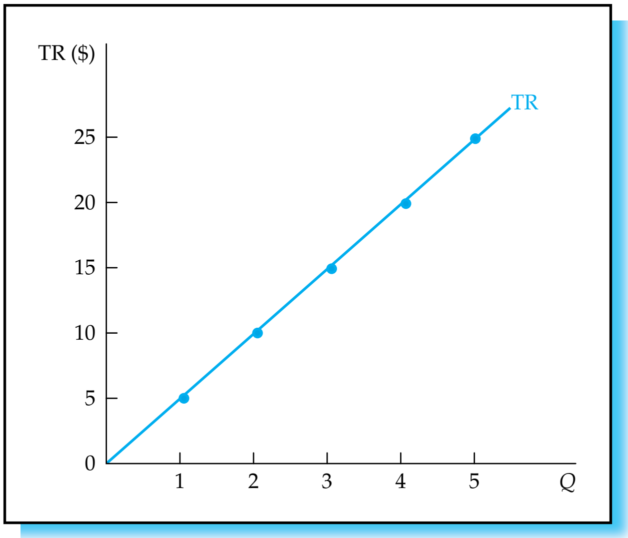


Figure 1 Total Revenue Function

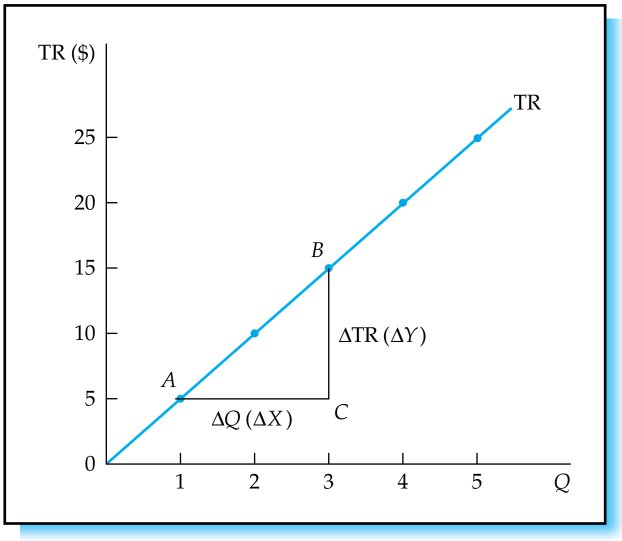


Figure 2 The Slope of the Total Revenue Line

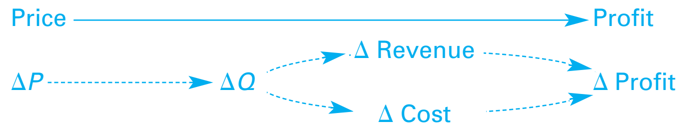
(i.e., $(\Delta Y)/(\Delta X)$). In other words, it represents the slope of the line plotted on the basis of Equation 2. The slope of a line is a measure of its steepness. This can be seen in Figure 1, which for purposes of discussion we have reproduced in Figure 2. In this figure, the steepness of the line between points *A* and *B* can be seen as *BC*/*AC*.

The slope of a function is critical to economic analysis because it shows the change in a dependent variable relative to a change in a designated independent variable. As explained in the next section of this appendix, this is the essence of *marginal analysis*.

The Importance of Marginal Analysis in Economic Analysis

One of the most important contributions that economic theory has made to managerial decision making is the application of what economists call *marginal analysis*. Essentially, marginal analysis involves the consideration of changes in the values of variables from some given starting point. Stated in a more formal mathematical manner, marginal analysis can be defined as the amount of change in a dependent variable that results from a unit change in an independent variable. If the functional relationship between the dependent and independent variables is linear, this change is represented by the slope of the line. In the case of our total revenue function, $\text{TR} = 5Q$, we can readily see that the coefficient, 5, indicates the marginal relationship between TR, the dependent variable, and Q, the independent variable. That is, total revenue is expected to change by $5 for every unit change in Q.

Most economic decisions made by managers involve some sort of change in one variable relative to a change in other variables. For example, a firm might want to consider raising or lowering the price from $5 per unit. Whether it is desirable to do so would depend on the resulting change in revenues or profits. Changes in these variables would in turn depend on the change in the number of units sold as a result of the change in price. Using the usual symbol for change, Δ (delta), this pricing decision can be summarized via the following illustration:



The top line shows that a firm's price affects its profit. The dashed arrows indicate that changes in price will change profit via changes in the number of units sold (ΔQ), in revenue, and in cost. Actually, each dashed line represents a key function used in economic analysis. ΔP to ΔQ represents the demand function. ΔQ to ΔRevenue is the total revenue function, ΔQ to ΔCost is the cost function, and ΔQ to ΔProfit is the profit function. Of course, the particular behavior or pattern of profit change relative to changes in quantity depends on how revenue and cost change relative to changes in Q.

Marginal analysis comes into play even if a product is new and being priced in the market for the first time. When there is no starting or reference point, different values of a variable may be evaluated in a form of sensitivity or what-if analysis. For example, the decision makers in a company such as IBM may price the company's new workstations by charting a list of hypothetical prices and then forecasting how many units the company will be able to sell at each price. By shifting from price to price, the decision makers would be engaging in a form of marginal analysis.[3]

Many other economic decisions rely on marginal analysis, including the hiring of additional personnel, the purchase of additional equipment, or a venture into a new line of business. In each case, it is the change in some variable (e.g., profit, cash flow, productivity, or cost) associated with the change in a firm's resource allocation that is of importance to the decision maker. Consideration of changes in relation to some reference point is also referred to as *incremental analysis.* A common distinction made between incremental and marginal analysis is that the former simply considers the change in the dependent variable, whereas the latter considers the change in the dependent variable relative to a one-unit change in the independent variable. For example, suppose lowering the price of a product results in a sales increase of 1,000 units and a revenue increase of $2,000. The incremental revenue would be $2,000, and the marginal revenue would be $2 ($2,000/1,000).

Functional Forms: A Variation on a Theme

For purposes of illustration, we often rely on a linear function to express the relationship among variables. This is particularly the case in chapter 3, on supply and demand. But there are many instances when a linear function is not the proper expression for changes in the value of a dependent variable relative to changes in some independent variable. For example, if a firm's total revenue does not increase

[3]One of the authors was a member of the pricing department of IBM for a number of years. This type of sensitivity analysis involving marginal relationships is indeed an important part of the pricing process.

at the same rate as additional units of its product are sold, a linear function is clearly not appropriate. To illustrate this phenomenon, let us assume a firm has the power to set its price at different levels and that its customers respond to different prices on the basis of the following schedule:

P	Q
$7	0
6	100
5	200
4	300
3	400
2	500
1	600
0	700

The algebraic and graphical expressions of this relationship are shown in Figure 3. As implied in the schedule and as shown explicitly in Figure 3, we assume a linear relationship between price and quantity demanded.

Based on the definition of total revenue as TR $= P \times Q$, we can create a total revenue schedule, as well as a total revenue equation and graph. These are shown in Figure 4.

Because we know that the demand curve is $Q = 700 - 100P$ and TR $= P \times Q$, we can arrive at the values of the coefficient and intercept terms, as well as the functional form in a very straightforward manner. First, we need to express P in terms of Q so we can substitute this relationship into the total revenue equation:

$$Q = 700 - 100P \tag{3}$$

or

$$P = 7 - 0.01Q \tag{4}$$

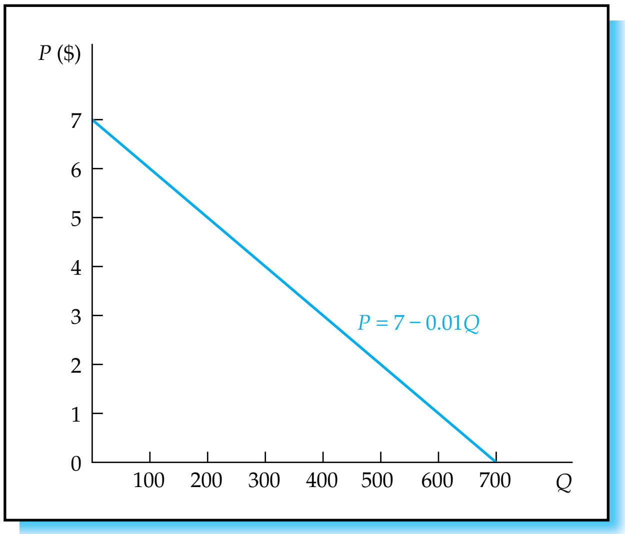


Figure 3 Demand Curve

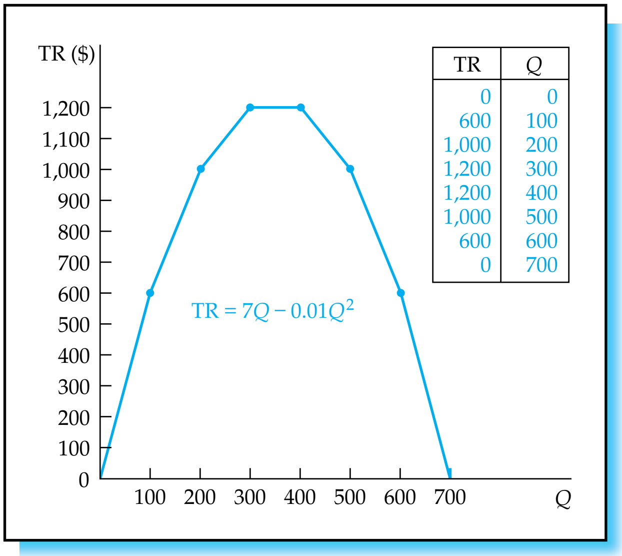


Figure 4 Total Revenue

Substituting the Equation 4 into the total revenue equation gives

$$\begin{aligned} \text{TR} &= P \times Q \\ &= (7 - 0.01Q)Q \\ &= 7Q - 0.01Q^2 \end{aligned} \tag{5}$$

As can be seen, a linear demand function results in a nonlinear total revenue function. More precisely, the functional relationship between total revenue and quantity seen here is expressed as a quadratic equation. Basically, this particular functional relationship is obtained whenever the independent variable is raised to the second power (i.e., squared) and to the first power. Graphically, quadratic equations are easily recognized by their parabolic shape. The parabola's actual shape and placement on the graph depend on the values and signs of the coefficient and intercept terms. Figure 5 shows four different quadratic functions.

If, in addition to being squared, the independent variable is raised to the third power (i.e., cubed), the relationship between the dependent and independent variables is called a cubic function. Figure 6 illustrates different cubic functions. As in the case of quadratic equations, the pattern and placement of these curves depend on the values and signs of the coefficients and intercept terms.

The independent variable can also be raised beyond the third power. However, anything more complex than a cubic equation is generally not useful for describing the relationship among variables in managerial economics. Certainly, there is no need to go beyond the cubic equation for purposes of this text. As readers see in ensuing chapters, the most commonly used forms of key functions are (1) linear demand function, (2) linear or quadratic total revenue function, (3) cubic production function, (4) cubic cost function, and (5) cubic profit function. Some variations to these relationships are also used, depending on the specifics of the examples being discussed.

There are other nonlinear forms used in economic analysis besides those just listed. These forms involve the use of exponents, logarithms, and reciprocals of the independent variables. Simple examples of these types of nonlinear form are shown in Figure 7. These forms are generally used in the statistical estimation of economic

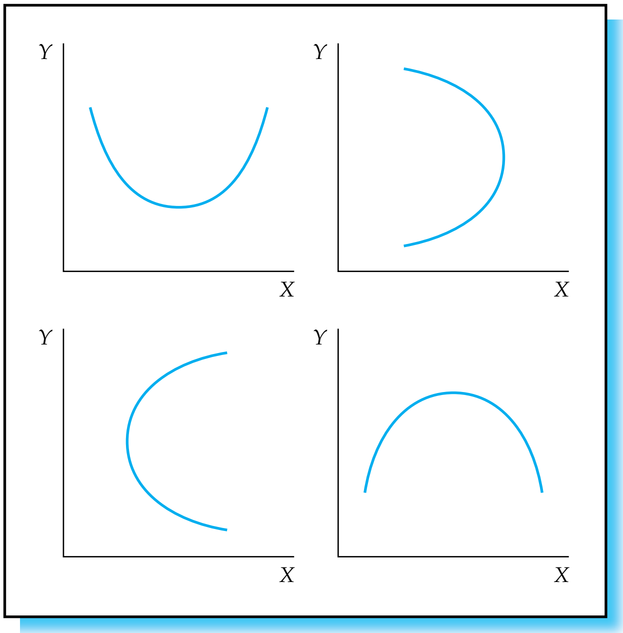


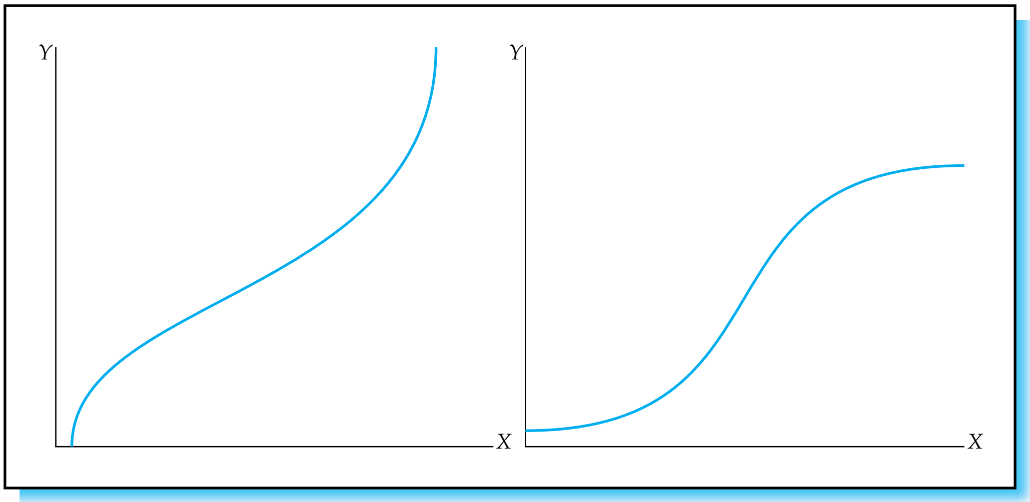
Figure 5 Examples of Quadratic Functions

functions, such as the demand, production, and cost functions, and in the forecasting of variables based on some trend over time (i.e., time series analysis). More is said about these particular functional forms in chapters 5, 8, and 9.

Continuous Functional Relationships

In plotting a functional relationship on a graph, we assume changes in the value of the dependent variable are related in a continuous manner to changes in the independent variable. Intuitively, a function can be said to be continuous if it can be

Figure 6 Examples of Cubic Functions



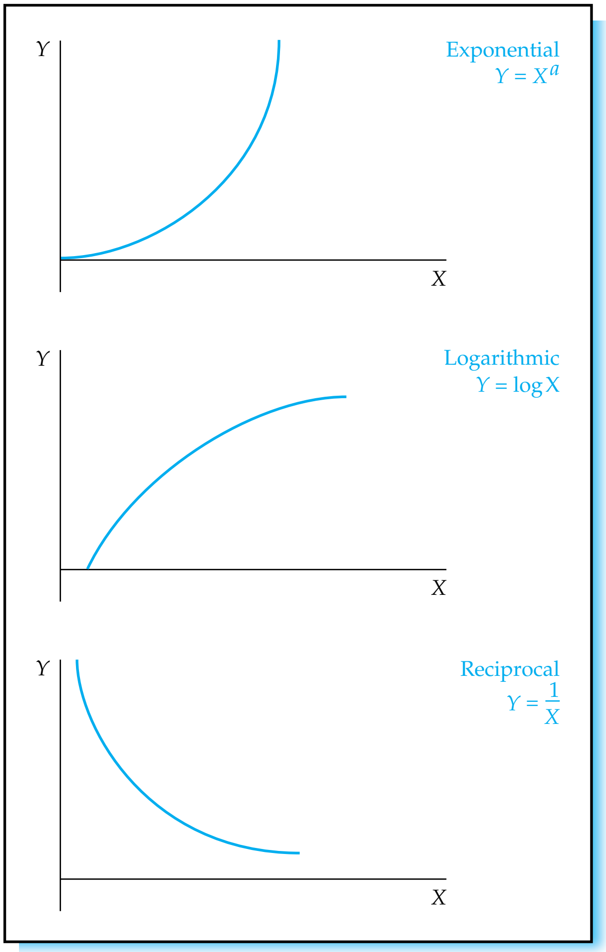


Figure 7 Examples of Selected Nonlinear Functions

drawn on a graph without taking the pencil off the paper.[4] Perhaps the best way to understand a continuous function is to observe its opposite, a function with discontinuity. Suppose the admission price to an amusement park is established as follows: ages 1 through 12 must pay \$3, ages 13 through 60 must pay \$8, and ages 61 and older must pay \$5. A graph of the relationship between admission price and age is shown in Figure 8. Notice that there is a jump or break in the graph at the level separating children from adults and the level separating adults from senior citizens. Because of these breaks in the relationship between the independent and dependent variables, this discontinuous relationship is also referred to as a *step function.*

Unless otherwise specified, the functional relationships analyzed in this text are considered to be continuous. Looking back at our example of the demand and total revenue functions, we can see that they indeed indicate a continuous relationship between price and quantity and between total revenue and quantity (see Figures 1 and 4). However, a closer look at the intervals used in the examples might lead you to question the applicability of a continuous function in actual business situations. For instance, let us observe again in Figure 9 the relationship between total revenue and quantity first shown in Figure 4.

[4]This particular way of explaining a continuous function is taken from Gulati, *College Mathematics.* To be sure, the author provides a much more rigorous definition of this concept.

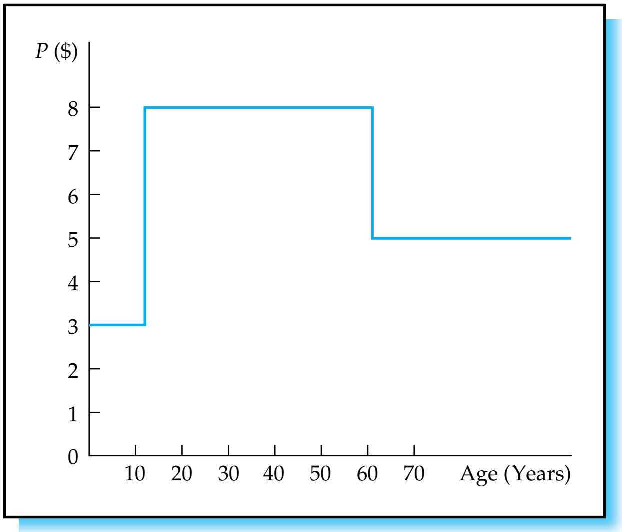


Figure 8 Example of a Step Function: Age Groups and Admission Price

The inquiring reader might ask whether this relationship, TR = $7Q - 0.01Q^2$, is in fact valid for points *within* each given interval. For example, if the firm sold 150 units, would it earn $825? Even if the answer is affirmative, to be a truly continuous function, the relationship would have to hold no matter how small the intervals of quantity considered.[5] For example, if the firm sold 150.567 units, its revenue would be $827.265.

But at this point, an adjustment to strict mathematics must be tempered with common sense. There are many instances in economic analysis in which continuous functions are assumed to represent relationships among variables, even though the variables themselves are subject to limitations in how finely they can be subdivided. For example, a firm may only be able to sell its product in lots of 100 or, at the very least, in single units. A firm might not want to consider price changes in terms of cents

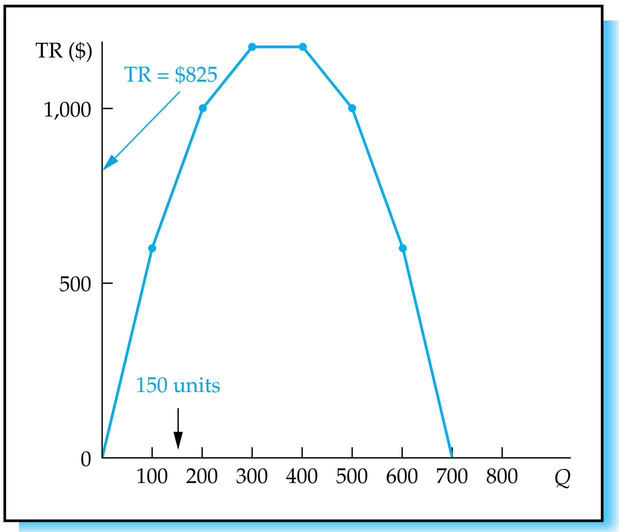


Figure 9 Discrete Intervals in a Continuous Function: The Example of Total Revenue

[5]According to mathematicians, "A function is said to be continuous over an open interval if it is continuous at *every* point in that interval" (Gulati, *College Mathematics*, p. 505).

but only in terms of dollars. In other cases, it might not be a matter of a firm's choice but of what resources are available. For example, suppose we have a function relating persons hired to the output they produce. (In chapter 6, this is referred to as the short-run production function.) Let us further suppose that labor resources are measured in terms of units of people (as opposed to hours, minutes, or even seconds of work time). Many economic activities in business involve variables that must be measured in discrete intervals (e.g., people, units of output, monetary units, machines, factories). For purposes of analysis, we assume all the economic variables are related to each other in a continuous fashion but are valid only at stated discrete intervals.

USING CALCULUS

Calculus is a mathematical technique that enables one to find instantaneous rates of change of a continuous function. That is, instead of finding the rate of change between two points on a plotted line, as shown in Figure 2, calculus enables us to find the rate of change in the dependent variable relative to the independent variable at a particular point on the function. However, calculus can be so applied *only* if a function is continuous. Thus, we needed to establish firmly the validity of using continuous functions to represent the relationships among economic variables.

Our brief introduction to calculus and its role in economic analysis begins with the statement that if all functional relationships in economics were linear, there would be no need for calculus! This point may be made clearer by referring to an intuitive definition of calculus. To quote the author of an extremely helpful and readable book on this subject:

> *Calculus, first of all, is wrongly named. It should never have been given that name. A far truer and more meaningful name is "SLOPE-FINDING."*[6]

It is not difficult to find the slope of a linear function. We simply take any two points on the line and find the change in Y relative to the change in X. The relative change is, of course, represented by the b coefficient in the linear equation. Moreover, because it is linear, the slope or rate of change remains the same between any two points over the entire range of intervals one wants to consider for the function. This is shown in the algebraic expression of the linear function by the constancy of the b coefficient.

However, finding the slope of a nonlinear function poses a problem. Let us arbitrarily take two points on the curve shown in Figure 10 and label them A and D. The slope or rate of change of Y relative to the change in X can be seen as DL/AL. Now, on this same curve let us find the slope of a point closer to point D and call it C. Notice that the slope of the line between these two points is less than the slope between D and A. (The measure of this slope is DM/CM.) The same holds true if we consider point B, a point that is still closer to D; the slope between B and D is less than the two slopes already considered. In general, we can state that in reference to the curve shown in Figure 10, the slope between point D and a point to the left decreases as the point moves closer to D. Obviously, this is not the case for a linear equation because the slope is constant.

[6]Eli S. Pine, *How to Enjoy Calculus*, Hasbrouck Heights, NJ: Steinlitz-Hammacher, 1983. Pine's definition is, of course, a simplification because it leaves out integral calculus. Nevertheless, we recommend this book highly for those who want a "user-friendly" review of differential calculus.

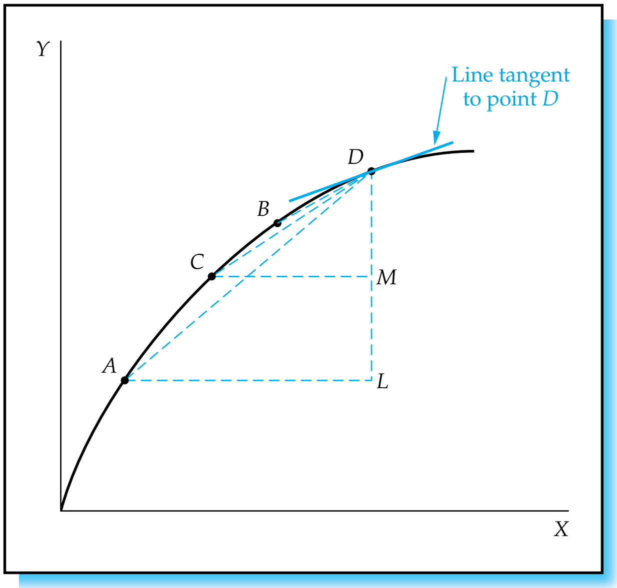


Figure 10 Finding the Slope of a Nonlinear Function

To understand how calculus enables us to find the slope or rate of change of a nonlinear function, let us resume the experiment. Suppose we keep on measuring changes in Y relative to smaller and smaller changes in X. Graphically, this can be represented in Figure 10 by moving point B toward point D. As smaller and smaller changes in X are considered, point B moves closer and closer to point D until the limit at which it appears to become one and the same with point D. When this occurs, the slope or rate of change of Y relative to X can be represented as point D itself. Graphically, this is represented by the slope of a line tangent to point D. In effect, this slope is a measure of the change in Y relative to a very small (i.e., infinitesimally small) change in X. To find the magnitude of the slope of tangency to any point on a line, we need to employ calculus, or more specifically, a concept used in calculus called the *derivative.*

In mathematics, a derivative is a measure of the change in Y relative to a very small change in X. Using formal mathematical notation, we can define the derivative as

$$\frac{dY}{dX} = \lim_{\Delta X \to 0} \frac{\Delta Y}{\Delta X}$$

This notation can be expressed as, "The derivative of Y with respect to X equals the limit (if such a limit exists) of the change in Y relative to the change in X as the change in X approaches zero."[7] As you can see from the discussion in the previous two paragraphs, the derivative turns out to be the slope of a line that is tangent to some given point on a curve. By convention, mathematicians use d to represent very small changes in a variable. Hence, dY/dX means "changes in Y relative to very small changes in X." For changes between two distinct points, the delta sign (Δ) is used.

[7]The concept of *limit* is critical to understanding the derivative. We tried to present an intuitive explanation of this concept by considering the movement of point B in Figure 10 closer and closer to point D, so that in effect the changes in X become smaller and smaller. *At the limit,* B becomes so close to D that, for all intents and purposes, it is the same as D. This situation would represent the smallest possible change in X. For a more formal explanation of *limit,* readers should consult any introductory calculus text.

Finding the Derivatives of a Function

There are certain rules for finding the derivatives of a function. We present in some detail two rules that are used extensively in this text. The other rules and their use in economic analysis are only briefly mentioned. Formal proofs of all these rules are not provided. Interested students may consult any introductory calculus text for this information.

Constants

The derivative of a constant must always be zero. Derivatives involve rates of change, and a constant, by definition, never changes in value. Expressed formally, if Y equals some constant (e.g., $Y = 100$), then

$$\frac{dY}{dX} = 0$$

The null value of the derivative of a constant is illustrated in Figure 11. Here we have assumed Y to have a constant value of 100. Clearly, this constant value of Y is unaffected by changes in the value of X. Thus, $dY/dX = 0$.

Power Functions

A *power function* is one in which the independent variable, X, is raised to the power of one or more. This type of function can be expressed in general terms as

$$Y = bX^n \tag{6}$$

where Y = Dependent variable
b = Coefficient of the dependent variable
X = Independent variable
n = Power to which the independent variable is raised

The rule for finding the derivative of this type of function is

$$\frac{dY}{dX} = nbX^{(n-1)} \tag{7}$$

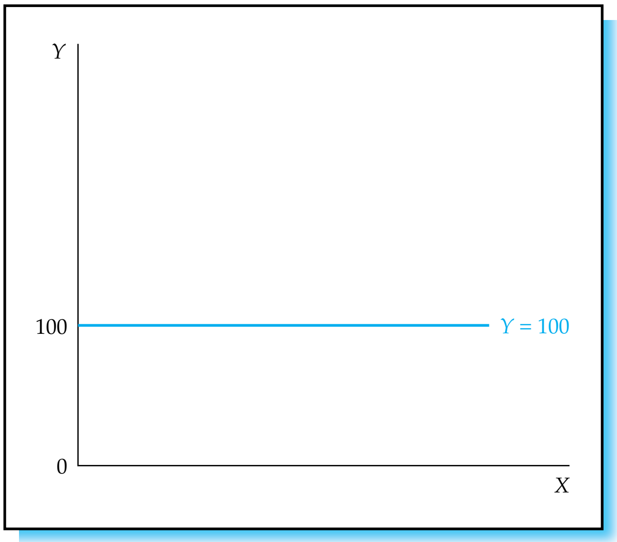


Figure 11 The Derivative of a Constant Equals Zero

Thus, suppose we have the equation

$$Y = 10X^3 \tag{8}$$

The derivative of this equation according to this rule is

$$\begin{aligned}\frac{dY}{dX} &= 3 \times 10X^{(3-1)} \\ &= 30X^2\end{aligned} \tag{9}$$

Thus, at the point where $X = 5$, the "instantaneous" rate of change of Y with respect to X is $30(5)^2$, or 750.

Sums and Differences

For convenience in presenting the rules in the remainder of this appendix, we use the following notations:

$U = g(X)$, where U is an unspecified function, g, of X
$V = h(X)$, where V is an unspecified function, h, of X

Given the function $Y = U + V$, the derivative of the sum (difference) is equal to the sum (difference) of the derivatives of the individual terms. In notational form,

$$\frac{dY}{dX} = \frac{dU}{dX} + \frac{dV}{dX}$$

For example, if $U = g(X) = 3X^2$, $V = h(X) = 4X^3$ and $Y = U + V = 3X^2 + 4X^3$, then

$$\frac{dY}{dX} = 6X + 12X^2$$

Products

Given the function $Y = UV$, its derivative can be expressed as follows:

$$\frac{dY}{dX} = U\frac{dV}{dX} + V\frac{dU}{dX}$$

This rule states that the derivative of the product of two expressions (U and V) is equal to the first term multiplied by the derivative of the second, plus the second term times the derivative of the first. For example, let $Y = 5X^2(7 - X)$. By letting $U = 5X^2$ and $V = (7 - X)$, we obtain

$$\begin{aligned}\frac{dY}{dX} &= 5X^2\frac{dV}{dX} + (7 - X)\frac{dU}{dX} \\ &= 5X^2(-1) + (7 - X)10X \\ &= -5X^2 + 70X - 10X^2 \\ &= 70X - 15X^2\end{aligned}$$

Quotients

Given the function $Y = U/V$, its derivative can be expressed as follows:

$$\frac{dY}{dX} = \frac{V(dU/dX) - U(dV/dX)}{V^2}$$

For example, suppose we have the following function:

$$Y = \frac{5X - 9}{10X^2}$$

Using the formula and letting $U = 5X - 9$ and $V = 10X^2$, we obtain the following:

$$\begin{aligned}
\frac{dY}{dX} &= \frac{10X^2 \times 5 - (5X - 9)20X}{100X^4} \\
&= \frac{50X^2 - 100X^2 + 180X}{100X^4} \\
&= \frac{180X - 50X^2}{100X^4} \\
&= \frac{18 - 5X}{10X^3}
\end{aligned}$$

Applying the Rules to an Economic Problem and a Preview of Other Rules for Differentiating a Function

There are several other rules for differentiating a function that are used in economic analysis. These involve differentiating a logarithmic function and the "function of a function" (often referred to in mathematics as the *chain rule*). We present these rules as they are needed in the appropriate chapters. In fact, almost all the mathematical examples involving calculus require only the rules for constants, powers, and sums and differences. As an example of how these three rules are applied, we return to the total revenue and demand functions presented earlier in this appendix. Recall that

$$\text{TR} = 7Q - 0.01Q^2 \tag{10}$$

Using the rules for powers and for sums and differences, we find that the derivative of this *function* is

$$\frac{d\text{TR}}{dQ} = 7 - 0.02Q \tag{11}$$

The derivative of the total revenue function is also called the *marginal revenue function* and plays an important part in many aspects of economic analysis. (See chapter 4 for a complete discussion of the definition and uses of marginal revenue.)

Turning now to the demand function first presented in Figure 3, we recall that

$$Q = 700 - 100P \tag{12}$$

Using the rules for constants, powers, and sums and differences, we see that the derivative of this function is:

$$\begin{aligned}
\frac{dQ}{dP} &= 0 - 1(100)P^{(1-1)} \\
&= -100P^0 \\
&= -100
\end{aligned} \tag{13}$$

Notice that, by the conventions of mathematical notation, variables such as P that have no stated exponent are assumed to be raised to the first power (i.e., $n = 1$).

Thus, based on the rule for the derivative of a power function $(n - 1)$ becomes $(1 - 1)$, or zero.[8] Therefore, dQ/dP is equal to the constant value 100. Recall our initial statement that there is no need for calculus if only linear functions are considered. This is supported by the results shown in Equation 13. Here we can see that the first derivative of the linear demand equation is simply the value of the b coefficient, 100. That is, no matter what the value of P, the change in Q with respect to a change in P is 100 (i.e., the slope of the linear function, or the b coefficient in the linear equation). Another way to express this is that for a linear function, there is no need to take the derivative of dY/dX. Instead, we can use the slope of the line represented by $\Delta Y/\Delta X$.

Partial Derivatives

Many functional relationships in the text entail a number of independent variables. For example, let us assume a firm has a demand function represented by the following equation:

$$Q = -100P + 50I + P_s + 2N \tag{14}$$

where Q = Quantity demanded
P = Price of the product
I = Income of customers
P_s = Price of a substitute product
N = Number of customers

If we want to know the change in Q with respect to a change in a particular independent variable, we can take the partial derivative of Q with respect to that variable. For example, the impact of a change in P on Q, with other factors held constant, would be expressed as

$$\frac{\delta Q}{\delta P} = -100 \tag{15}$$

The conventional symbol used in mathematics for the partial derivative is the lowercase Greek letter delta, δ. Notice that all we did was use the rule for the derivative of a power function on the P variable. Because the other independent variables, I, P_s, and N, are held constant, they are treated as constants in taking the partial derivative. (As you recall, the derivative of a constant is zero.) Thus, the other terms in the equation drop out, leaving us with the instantaneous impact of the change in P on Q. This procedure applies regardless of the powers to which the independent variables are raised. It just so happens that in this equation, all the independent variables are raised only to the first power.[9]

Finding the Maximum and Minimum Values of a Function

A primary objective of managerial economics is to find the optimal values of key variables. This means finding "the best" possible amount or value under certain circumstances. Marginal analysis and the concept of the derivative are very helpful in finding optimal values. For example, given a total revenue function, a firm might

[8]The rule in algebra is that any value raised to the zeroth power is equal to unity.
[9]An expression such as this is referred to in mathematics as a *linear additive equation*.

want to find the number of units it must sell to maximize its revenue. Taking the total revenue function first shown in Equation 5, we have

$$\mathrm{TR} = 7Q - 0.01Q^2 \tag{16}$$

The derivative of this function (i.e., marginal revenue) is

$$\frac{d\mathrm{TR}}{dQ} = 7 - 0.02Q \tag{17}$$

Setting the first derivative of the total revenue function (or the marginal revenue function) equal to zero and solving for the revenue-maximizing quantity, Q^*, gives us[10]

$$\begin{aligned} 7 - 0.02Q &= 0 \\ Q^* &= 350 \end{aligned} \tag{18}$$

Thus, the firm should sell 350 units of its product if it wants to maximize its total revenue. In addition, if the managers want to know the price that the firm should charge to sell the "revenue-maximizing" number of units, they can go back to the demand equation from which the total revenue function was derived, that is,

$$P = 7 - 0.01\,Q \tag{19}$$

By substituting the value of Q^* into this equation, we obtain

$$\begin{aligned} P^* &= 7 - 0.01(350) \\ &= \$3.50 \end{aligned} \tag{20}$$

The demand function, the total revenue function, and the revenue-maximizing price and quantity are all illustrated in Figure 12.

To further illustrate the use of the derivative in finding the optimum, let us use an example that plays an important part in chapters 8 and 9. Suppose a firm wants to find the price and output levels that will maximize its profit. If the firm's revenue and cost functions are known, it is a relatively simple matter to use the derivative of these functions to find the optimal price and quantity. To begin with, let us assume the following demand, revenue, and cost functions:

$$Q = 17.2 - 0.1P \tag{21}$$

or

$$P = 172 - 10Q \tag{22}$$

$$\mathrm{TR} = 172Q - 10Q^2 \tag{23}$$

$$\mathrm{TC} = 100 + 65Q + Q^2 \tag{24}$$

By definition, profit (π) is equal to total revenue minus total cost. That is,

$$\pi = \mathrm{TR} - \mathrm{TC} \tag{25}$$

[10]Henceforth, all optimal values for Q and P (e.g., values that maximize revenue or profit or minimize cost) are designated with an asterisk.

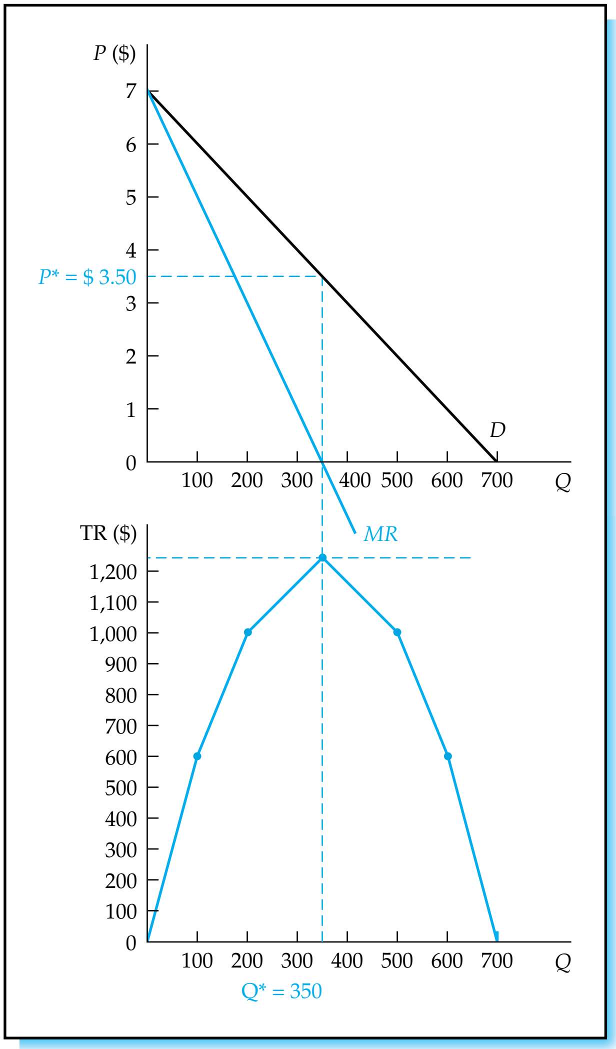


Figure 12 Demand Function, Total Revenue Function, and Revenue-Maximizing Price and Quantity

Substituting Equations 23 and 24 into 25 gives us:

$$\begin{aligned} \pi &= 172Q - 10Q^2 - 100 - 65Q - Q^2 \\ &= -100 + 107Q - 11Q^2 \end{aligned} \tag{26}$$

To find the profit-maximizing output level, we simply follow the same procedure used to find the revenue-maximizing output level. We take the derivative of the total profit function, set it equal to zero, and solve for Q^*:

$$\begin{aligned} \frac{d\pi}{dQ} &= 107 - 22Q = \\ 22Q &= 107 \end{aligned} \tag{27}$$

$$Q^* = 4.86 \tag{28}$$

The total revenue and cost functions and the total profit function are illustrated in Figures 13*a* and 13*b*, respectively.

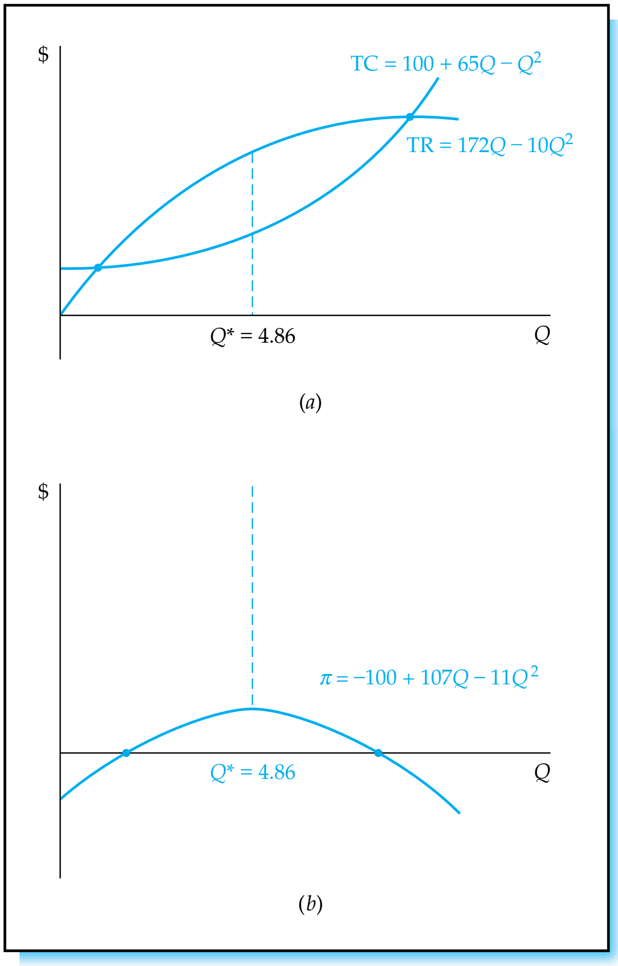


Figure 13 Total Revenue, Total Cost, and Total Profit Functions

Distinguishing Maximum and Minimum Values in the Optimization Problem

In economic analysis, finding the optimum generally means finding either the maximum or minimum value of a variable, depending on what type of function is being considered. For example, if the profit or total revenue function is the focus, the maximum value is obviously of interest. If a cost function is being analyzed, its minimum value would be the main concern. Taking the derivative of a function, setting it equal to zero, and then solving for the value of the independent variable enables us to find the maximum or minimum value of the function.

However, there may be instances in which a function has both a maximum and a minimum value. When this occurs, the method described previously cannot tell us whether the optimum is a maximum or a minimum. This situation of indeterminacy can be seen in Figure 14*a*. The graph in this figure represents a cubic function of the general form $Y = a - bX + cX^2 - dX^3$. Clearly, this function has both a minimum (point A) and a maximum (point C). As a generalization, if we took the first derivative at the four points designated in Figure 14*a*, we would find that

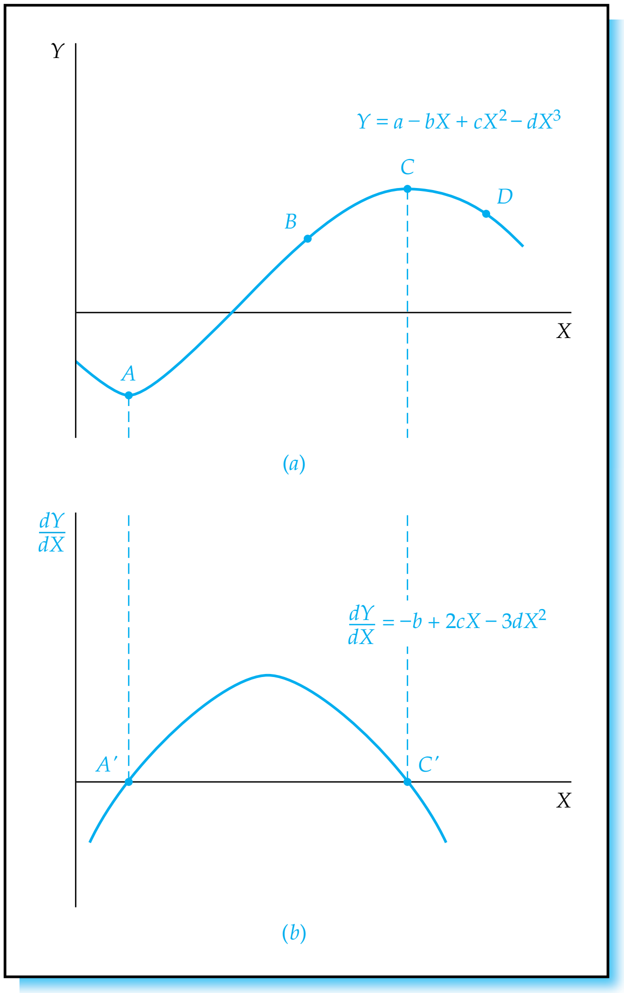


Figure 14 Maximum and Minimum Points of a Cubic Function

At point A, $dY/dX = 0$.
At point B, $dY/dX > 0$.
At point C, $dY/dX = 0$.
At point D, $dY/dX < 0$.

As expected, the first derivatives of points A and C are equal to zero, reflecting the fact that their lines of tangency are horizontal (i.e., have zero slope). The positive and negative values of the derivatives at points B and D are reflective of their respective upward and downward lines of tangency. However, because *both* points A and C have first derivatives equal to zero, a problem arises if we want to know whether these points indicate maximum or minimum values of Y. Of course, in Figure 14 we can plainly see that point A is the minimum value and point C is the maximum value. However, there is a formal mathematical procedure for distinguishing a function's maximum and minimum values. This procedure requires the use of a function's *second derivative.*

The second derivative of a function is the derivative of its first derivative. The procedure for finding the second derivative of a function is quite simple. All the rules for finding the first derivative apply to obtaining the second derivative. Conceptually, we can consider the second derivative of a function as a measure of the rate of change of the first derivative. In other words, it is a measure of "the rate of change of the rate of change."[11]

Let us illustrate precisely how the second derivative is used to determine the maximum and minimum values by presenting a graph of the function's first derivative in Figure 14*b.* As a check on your understanding of this figure, notice that, as expected, the second derivative has a negative value when the original function decreases in value and a positive value when the original function increases in value. But now consider another aspect of this figure. Recall that the second derivative is a measure of the rate of change of the first derivative. Thus, graphically, we can find the second derivative by evaluating the slopes of lines tangent to points on the graph of the *first derivative.* See points A' and C' in Figure 14*b.* By inspection of this figure, it should be quite clear that the slope of the line tangent to point A' is positive and the slope of the line tangent to point C' is negative. This enables us to conclude that at the minimum point of a function, the second derivative is *positive.* At the maximum point of a function, the second derivative is *negative.*[12]

Using mathematical notation, we can now state the *first-* and *second-order conditions* for determining the maximum or minimum values of a function.

Maximum value:	$dY/dX = 0$ (first-order condition)
	$d^2/dx^2 < 0$ (second-order condition)
Minimum value:	$dY/dX = 0$ (first-order condition)
	$d^2/dx^2 > 0$ (second-order condition)

We now illustrate how the first- and second-order conditions are used to find the profit-maximizing level of output for a firm. Suppose this firm has the following revenue and total cost functions:

$$\text{TR} = 50Q \tag{29}$$

$$\text{TC} = 100 + 60Q - 3Q^2 + 0.1Q^3 \tag{30}$$

Based on these equations, the firm's total profit function is

$$\begin{aligned}\pi &= 50Q - (100 + 60Q - 3Q^2 + 0.1Q^3)\\ &= 50Q - 100 - 60Q + 3Q^2 - 0.1Q^3\\ &= -100 - 10Q + 3Q^2 - 0.1Q^3\end{aligned} \tag{31}$$

[11]Mathematics and physics texts often use the example of a moving automobile to help distinguish the first and second derivatives. To begin with, the function can be expressed as $M = f(T)$, or miles traveled is a function of the time elapsed. The first derivative of this function describes the automobile's velocity. As an example of this, imagine a car moving at the rate of speed of 45 miles per hour. Now suppose this car has just entered a freeway, starts to accelerate, and then reaches a speech of 75 mph. The measure of this acceleration as the car goes from 45 mph to 75 mph is the second derivative of the function. Because the car is accelerating (i.e., going faster and faster), the second derivative is some positive value. In other words, the distance (measured in miles) that the car is traveling is increasing at an *increasing* rate. To extend this example a bit further, suppose the driver of the car, realizing that this section of the freeway is closely monitored by radar, begins to slow down to the legal speed limit of 60 mph. As the driver slows down, or decelerates, the speed at which the car is traveling is reduced. In other words, the distance (in miles) that the car is traveling is increasing at a *decreasing* rate. Deceleration, being the opposite of acceleration, implies that the second derivative in this case is negative.

[12]Let us return to the automobile example for an alternative explanation of the second-order condition for determining a function's maximum value. Let us imagine that, at the very moment the car reached 75 mph, the driver started to slow down. This means that at the precise moment the accelerating car started to decelerate, it had reached its *maximum* speed. In mathematical terms, if a function is increasing at an *increasing* rate (i.e., its second derivative is positive), then the moment it starts to increase at a *decreasing* rate (i.e., its second derivative becomes negative), it has reached its maximum value. Similar reasoning can be used to explain the second-order condition for determining the minimum value of a function.

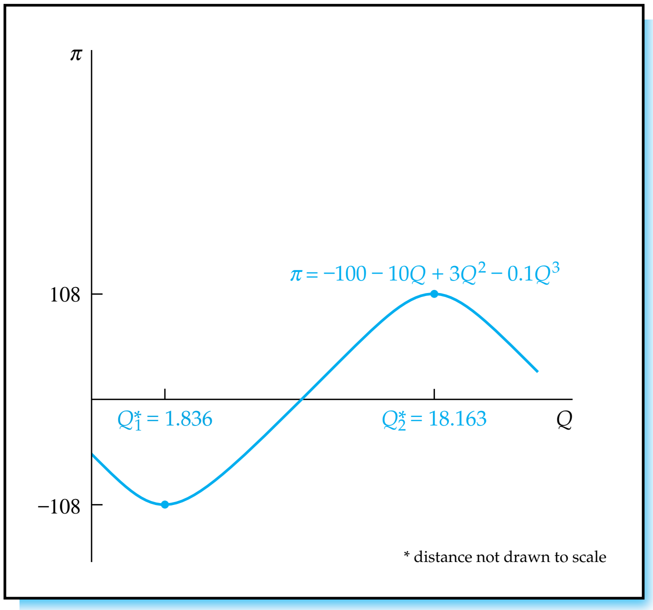


Figure 15 Cubic Profit Function

Notice that this firm's profit function contains a term that is raised to the third power because its cost function is also raised to the third power. In other words, the firm is assumed to have a cubic cost function, and therefore, it has a cubic profit function. Plotting this cubic profit function gives us the graph in Figure 15. We can observe in this figure that the level of output that maximizes the firm's profit is about 18.2 units, and the level of output that minimizes its profit (i.e., maximizes its loss) is about 1.8 units.

Let us employ calculus along with the first- and second-order conditions to determine the point at which the firm maximizes its profit. We begin as before by finding the first derivative of the profit function, setting it equal to zero, and solving for the value of Q that satisfies this condition.

$$\pi = -100 - 10Q + 30Q^2 - 0.1Q^3 \tag{32}$$

$$\begin{aligned} \frac{d\pi}{dQ} &= -10 + 6Q - 0.3Q^2 \\ &= -0.3Q^2 + 6Q - 10 = 0 \end{aligned} \tag{33}$$

Note that Equation 33 has been rearranged to conform to the general expression for a quadratic equation. Because the first derivative of the profit function is quadratic, there are two possible values of Q that satisfy the equation.[13]

$$Q_1^* = 1.836 \qquad Q_2^* = 18.163$$

As expected, Q_1^* and Q_2^* coincide with the two points shown in Figure 13. Although both Q_1^* and Q_2^* fulfill the first-order condition, only one satisfies the second-order condition. To see this, let us find the second derivative of the function by taking the derivative of the marginal profit function expressed in Equation 33:

$$\frac{d^2\pi}{dQ^2} = -0.6Q + 6$$

By substitution, we see that at output level 1.836 the value of the second derivative is a positive number:

$$\frac{d^2\pi}{dQ^2} = -0.6(1.836) + 6 = 4.89$$

In contrast, we see that at output level 18.163 the value of the second derivative is a negative number:

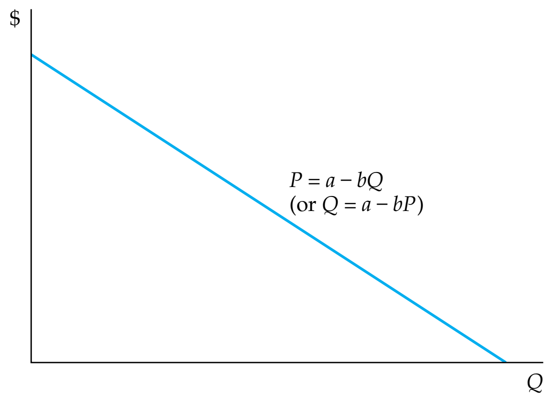
$$\frac{d^2\pi}{dQ^2} = -0.6(18.163) + 6 = -4.89$$

Thus, we see that only Q_2^* enables us to adhere to the second-order condition that $d^2\pi/dQ^2 < 0$. This confirms in a formal, mathematical manner what we already knew from plotting and evaluating the graph of the firm's total profit function: Q_2 is the firm's profit-maximizing level of output.

Five Key Functions Used in This Text

Five key functions are used in this text: (1) demand, (2) total revenue, (3) production, (4) total cost, and (5) profit. The following diagrams show the algebraic and graphical expressions for these functions. As can be seen, the demand function is linear, the total revenue function is quadratic, and the production, cost, and profit functions are cubic. Note that the last three functions all refer to economic conditions in the short run.

1. Demand



[13]The economic rationale for cost functions of different degrees is explained in chapter 7. The first derivative of a cubic function is a quadratic function. Such a function can be expressed in the general form

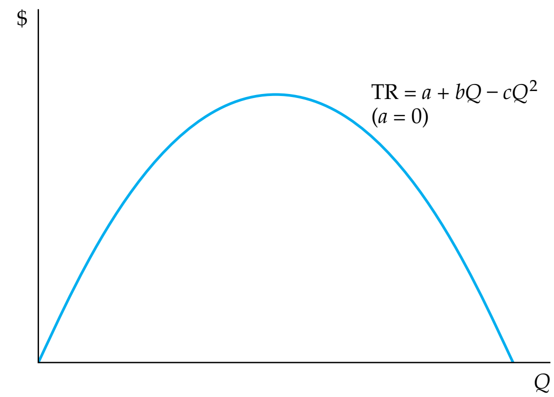
$$Y = aX^2 + bX + c$$

Perhaps you recall from your previous studies of algebra that the values of X that set the quadratic function equal to zero can be found by using the formula for the "roots" of a quadratic equation:

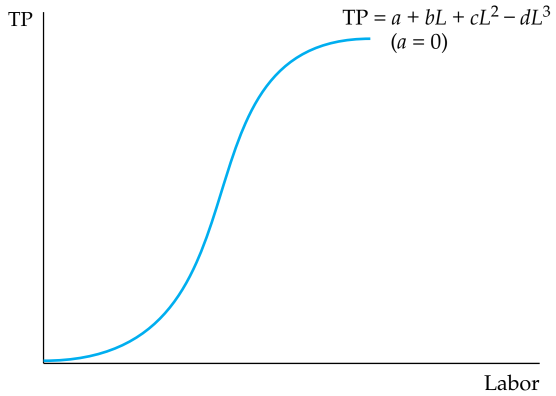
$$X = \frac{-b \pm \sqrt{b^3 - 4ac}}{2a}$$

By substituting the values of the coefficients in Equation 33 (i.e., $a = -0.3$, $b = 6$, $c = -10$), we obtain the answers shown.

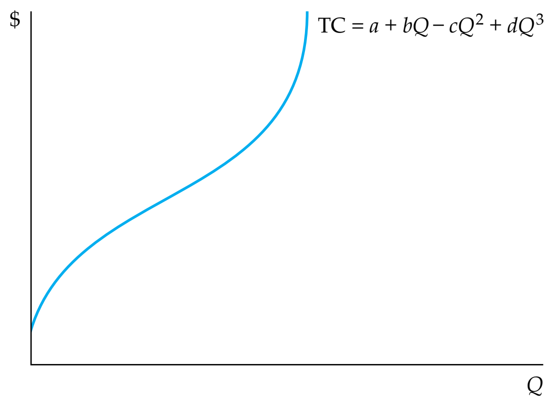
2. Total revenue



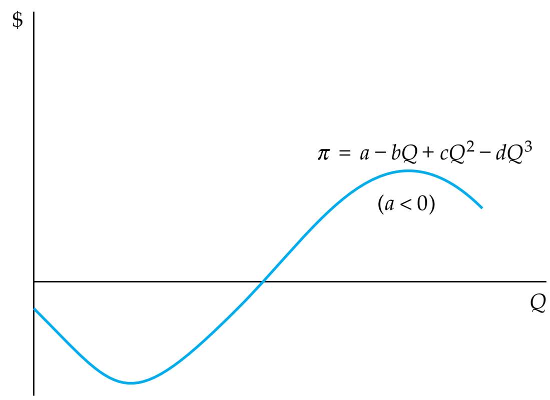
3. Production (short run)



4. Cost (short run)



5. Profit (short run)



SUMMARY

As you proceed with your study of managerial economics and the reading of the text, you will find that the essence of economic analysis is the study of functional relationships between certain dependent variables (e.g., quantity demanded, revenue, cost, profit) and one or more independent variables (e.g., price, income, quantity sold). Mathematics is a tool that can greatly facilitate the analysis of these functional relationships. For example, rather than simply saying that "the quantity of a product sold depends on its price," we can use an algebraic equation to state precisely how many units of a product a firm can expect to sell at a particular price. Moreover, when we engage in a marginal analysis of the impact of price on quantity demanded, we can use the first derivative of this equation to measure the change in quantity demanded relative to changes in price.[14] Furthermore, as shown in this appendix, the precise algebraic expression of the demand function enables us to derive a firm's total revenue and marginal revenue functions. With the help of calculus, the optimal price and quantity (e.g., the price and quantity that maximize revenue) can be quickly found.

The more data a firm is able to obtain about its key economic functions (i.e., demand, revenue, production, cost, and profit), the more mathematics can be employed in the analysis. The more that mathematics can be used, the more precise a manager can be about such key decisions as the best price to charge, the best markets to compete in, and the most desirable levels of resource allocation. Unfortunately, in the real world firms do not often have the luxury of accurate or complete data with which to work. This is another aspect of decision making and is discussed in chapter 5.

QUESTIONS

1. Define the following terms: *function, variable, independent variable, dependent variable, functional form.*
2. Briefly describe how a function is represented in tabular form, in graphical form, and in an equation. Illustrate using the relationship between price and quantity expressed in a demand function.
3. Express in formal mathematical terms the following functional relationships. You may use the general form $Y = f(X)$. However, in each case be as specific as possible about what variables are represented by Y and X. (For example, in the first relationship, advertising is the X variable and could be measured in terms of the amount of advertising dollars spent annually by a firm.)
   a. The effectiveness of advertising
   b. The impact on output resulting from increasing the number of employees
   c. The impact on labor productivity resulting from increasing automation
   d. The impact on sales and profits resulting from price reductions
   e. The impact on sales and profits resulting from a recession
   f. The impact on sales and profits resulting from changes in the financial sector (e.g., the stock market or the bond market)
   g. The impact on cost resulting from the use of outside vendors to supply certain components in the manufacturing process

[14]In chapter 4, you see how this first derivative is incorporated into a formula for *elasticity,* a value indicating the percentage change in a dependent variable, such as units sold, with a percentage change in an independent variable, such as price.

4. What is a continuous function? Does the use of continuous functions to express economic relationships present any difficulties in analyzing real-world business problems? Explain.
5. Define in mathematical terms the slope of a line. Why is the slope considered to be so important in the quantitative analysis of economic problems?
6. Define *marginal analysis.* Give examples of how this type of analysis can help a managerial decision maker. Are there any limitations to using this type of analysis in actual business situations? Explain.
7. Explain why the first derivative of a function is an important part of marginal analysis.
8. Explain how an analysis of the first derivative of the function $Y = f(X)$ enables one to find the point at which the Y variable is at its maximum or minimum.
9. (Optional) Explain how an analysis of the second derivative of a function enables one to determine whether the variable is a maximum or a minimum.
10. Briefly explain the difference between $\Delta Y/\Delta X$ and dY/dX. Explain why in a linear equation there is no difference between the two terms.

Problems

1. Answer the following questions on the basis of the accompanying demand schedule.

Price	Quantity
$100	25
80	35
60	45
40	55
20	65
0	75

   a. Express the schedule as an algebraic equation in which Q is the dependent variable. Plot this on a graph.
   b. Express the schedule as an algebraic equation in which P is the dependent variable. Plot this on a graph.

2. You are given the following demand equations:

$$Q = 450 - 16P$$
$$Q = 360 - 80P$$
$$Q = 1{,}500 - 500P$$

   a. Determine each equation's total revenue and marginal revenue equations.
   b. Plot the demand equation and the marginal and total revenue equations on a graph.
   c. Use calculus to determine the prices and quantities that maximize the revenue for each equation. Show the points of revenue maximization on the graphs that you have constructed.

3. You are given the following cost equations:

$$\text{TC} = 1{,}500 + 300Q - 25Q^2 + 1.5Q^3$$
$$\text{TC} = 1{,}500 + 300Q + 25Q^2$$
$$\text{TC} = 1{,}500 + 300Q$$

a. Determine each equation's average variable cost, average cost, and marginal cost.
b. Plot each equation on a graph. On separate graphs, plot each equation's average variable cost, average cost, and marginal cost.
c. Use calculus to determine the minimum point on the marginal cost curve.

4. Given the demand equation shown, perform the following tasks:

$$Q = 10 - .004P$$

a. Combine this equation with each cost equation listed in question 3. Use calculus to find the price that will maximize the short-run profit for each of the cost equations.
b. Plot the profit curve for each of the cost equations.